

Numerical Solution of an Inverse Electrocardiography Problem for a Medium with Piecewise Constant Electrical Conductivity

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Abstract—A numerical method is proposed for solving an inverse electrocardiography problem for a medium with a piecewise constant electrical conductivity. The method is based on the method of boundary integral equations and Tikhonov regularization.

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Key words: numerical solution of inverse electrocardiography problems, method of boundary integral equations, regularization method.

The inverse electrocardiography problem in the form of potentials consists in reconstructing the potential on the outer surface of the heart from the potential measured on the chest surface [1, 2]. The study of this problem is motivated by the clinical implementation of new treatment methods for cardiac arrhythmia.

Algorithms for solving the inverse electrocardiography problem for model geometry of the torso and the heart were described in [3]. Realistic geometry was used in [4], but the chest was assumed to be homogeneous. In this paper, the inverse electrocardiography problem is considered for realistic geometry of the torso and the heart with allowance for the internal inhomogeneity of the chest.

Consider a domain Ω in \mathbb{R}^3 with a closed outer boundary Γ_B and a closed inner boundary Γ_H (see Fig. 1). Γ_B is the union of two surfaces Γ_T and Γ_E . Two disjoint domains $\Omega_i \subset \Omega$ with boundaries Γ_i ($i = 1, 2$) are given in Ω . This geometric configuration is interpreted as follows: Γ_H is the surface of the heart, Γ_E is the part of the torso surface on which the electric field potential is measured, Γ_T denotes the upper and lower sections of the torso, and Ω_i ($i = 1, 2$) are inhomogeneity domains in the human chest (left and right lungs).

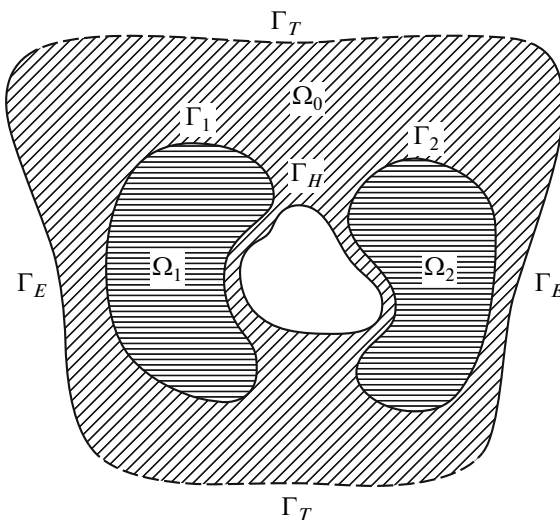


Fig. 1.

Define $\Omega_0 = \Omega \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_2)$, $\Gamma_0 = \Gamma_E$, and $\Gamma_3 = \Gamma_H \cup \Gamma_T$. Consider the following boundary value problem: find a function $u(x)$, $x \in \bar{\Omega}$ such that $u(x) = u_i(x)$, $x \in \bar{\Omega}_i$, $i = 0, 1, 2$,

$$\Delta u_i(x) = 0, \quad x \in \Omega_i, \quad i = 0, 1, 2, \tag{1}$$

$$u_0(x) = \varphi(x), \quad x \in \Gamma_3, \tag{2}$$

$$\frac{\partial u_0(x)}{\partial n} = 0, \quad x \in \Gamma_0, \tag{3}$$

$$u_0(x) = u_i(x), \quad x \in \Gamma_i, \quad i = 1, 2, \tag{4}$$

$$\sigma_0 \frac{\partial u_0(x)}{\partial n} = \sigma_i \frac{\partial u_i(x)}{\partial n}, \quad x \in \Gamma_i, \quad i = 1, 2. \tag{5}$$

Here, $\varphi(x)$ is a given function, σ_i ($i = 0, 1, 2$) are given positive constants, $\varphi(x)$ is the value of the potential on Γ_3 , and σ_i is a coefficient determining the electrical conductivity of the tissue occupying Ω_i .

Problem (1)–(5) defines an operator A that maps the values of the potential $\varphi(x)$ on Γ_3 to its values $u_0(x)$ on Γ_0 . The unique solvability of problem (1)–(5) and its numerical solution were considered in [5].

The inverse electrocardiography problem is formulated as follows. Find $u(x)$, $x \in \bar{\Omega}$ such that $u(x) = u_i(x)$, $x \in \bar{\Omega}_i$, $i = 0, 1, 2$,

$$\Delta u_i(x) = 0, \quad x \in \Omega_i, \quad i = 0, 1, 2, \tag{6}$$

$$u_0(x) = \psi(x), \quad x \in \Gamma_0, \tag{7}$$

$$\frac{\partial u_0(x)}{\partial n} = 0, \quad x \in \Gamma_0, \tag{8}$$

$$u_0(x) = u_i(x), \quad x \in \Gamma_i, \quad i = 1, 2, \tag{9}$$

$$\sigma_0 \frac{\partial u_0(x)}{\partial n} = \sigma_i \frac{\partial u_i(x)}{\partial n}, \quad x \in \Gamma_i, \quad i = 1, 2, \tag{10}$$

where $\psi(x)$ is a given function obtained by measurements on the torso surface.

Problem (6)–(10) is a generalization of the Cauchy problem for the Laplace equation, and it is an ill-posed problem. A major manifestation of its ill-posedness is that $u(x)$ is unstable with respect to small variations in $\psi(x)$.

Problem (6)–(10) can be reformulated as the problem of finding the values of $u(x)$ on Γ_3 , provided that $u(x)$ satisfies problem (6)–(10). The latter problem can also be written as an operator equation of the first kind

$$A\varphi = \psi, \tag{11}$$

where ψ is a given function, A is the operator defined by problem (1)–(5), and $\varphi(x)$ is an unknown function.

To find a numerical solution of the inverse problem, we use the method of boundary integral equations. Consider the domain Ω_0 with the boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Writing the third Green's identity in Ω_0 yields

$$2\pi u_0(x) = \int_{\Gamma} \frac{\partial u_0(y)}{\partial n} \frac{1}{|x-y|} ds_y - \int_{\Gamma} u_0(y) \frac{\partial}{\partial n} \frac{1}{|x-y|} ds_y, \quad x \in \Gamma, \tag{12}$$

where n is the outward unit normal to Ω_0 . This equality can be rewritten as

$$2\pi u_0(x) = \sum_{i=0}^3 \left(\int_{\Gamma_i} \frac{\partial u_0(y)}{\partial n} \frac{1}{|x-y|} ds_y - \int_{\Gamma_i} u_0(y) \frac{\partial}{\partial n} \frac{1}{|x-y|} ds_y \right), \quad x \in \Gamma. \tag{13}$$

Writing the third Green's identity in the domain Ω_1 with boundary Γ_1 , we obtain

$$-2\pi u_1(x) = \int_{\Gamma_1} \frac{\partial u_1(y)}{\partial n} \frac{1}{|x-y|} ds_y - \int_{\Gamma_1} u_1(y) \frac{\partial}{\partial n} \frac{1}{|x-y|} ds_y, \quad x \in \Gamma_1, \quad (14)$$

where n is the inward unit normal to Ω_1 . Using conditions (9) and (10) gives

$$-2\pi u_0(x) = \frac{\sigma_0}{\sigma_1} \int_{\Gamma_1} \frac{\partial u_0(y)}{\partial n} \frac{1}{|x-y|} ds_y - \int_{\Gamma_1} u_0(y) \frac{\partial}{\partial n} \frac{1}{|x-y|} ds_y, \quad x \in \Gamma_1. \quad (15)$$

Similarly, for the domain Ω_2 with boundary Γ_2 ,

$$-2\pi u_0(x) = \frac{\sigma_0}{\sigma_2} \int_{\Gamma_2} \frac{\partial u_0(y)}{\partial n} \frac{1}{|x-y|} ds_y - \int_{\Gamma_2} u_0(y) \frac{\partial}{\partial n} \frac{1}{|x-y|} ds_y, \quad x \in \Gamma_2. \quad (16)$$

Let v_l denote the values of u_0 on Γ_l , and let q_l denote the values of the normal derivative of u_0 on Γ_l . With this notation, formula (13) can be rewritten as

$$2\pi v_l(x) = \sum_{i=0}^3 \left(\int_{\Gamma_i} q_i(y) \frac{1}{|x-y|} ds_y - \int_{\Gamma_i} v_i(y) \frac{\partial}{\partial n} \frac{1}{|x-y|} ds_y \right), \quad x \in \Gamma_l, \quad l = 0, 1, 2, 3, \quad (17)$$

and formulas (15) and (16) become

$$-2\pi v_l(x) = \frac{\sigma_0}{\sigma_l} \int_{\Gamma_l} q_l(y) \frac{1}{|x-y|} ds_y - \int_{\Gamma_l} v_l(y) \frac{\partial}{\partial n} \frac{1}{|x-y|} ds_y, \quad x \in \Gamma_l, \quad l = 1, 2. \quad (18)$$

Each surface Γ_l is approximated by a polygonal surface $\hat{\Gamma}_l$ ($l = 0, 1, 2, 3$) consisting of plane triangles. The boundary of Ω_0 is approximated by the surface $\hat{\Gamma} = \hat{\Gamma}_0 \cup \hat{\Gamma}_1 \cup \hat{\Gamma}_2 \cup \hat{\Gamma}_3$. Introducing a unified numbering of the triangles on $\hat{\Gamma}$, we refer to each triangle as a boundary element ζ_k , $k = 1, 2, \dots, N$, where N is the total number of elements on $\hat{\Gamma}$. In each element ζ_k , the nodal point x_k ($k = 1, 2, \dots, N$) is defined as the barycenter of ζ_k .

On $\hat{\Gamma}$, we introduce a system of linearly independent compactly supported basis functions $\phi_j(x)$, $j = 1, 2, \dots, N$ defined as

$$\phi_j(x) = \begin{cases} 1, & x \in \zeta_j, \\ 0, & x \notin \zeta_j, \end{cases} \quad (19)$$

Let M_l and P_l ($l = 0, 1, 2, 3$) be numbers such that $x_j \in \hat{\Gamma}_l$ for $M_l \leq j \leq P_l$. The functions $v_l(x)$ and $q_l(x)$ are approximately represented as expansions in terms of $\phi_j(x)$:

$$\tilde{v}_l(x) = \sum_{j=M_l}^{P_l} \alpha_j \phi_j(x), \quad \tilde{q}_l(x) = \sum_{j=M_l}^{P_l} \beta_j \phi_j(x), \quad l = 0, 1, 2, 3. \quad (20)$$

Writing discrete analogues of formulas (17) and (18) for each nodal point x_k gives

$$2\pi \tilde{v}_l(x_k) = \sum_{i=0}^3 \left(\int_{\hat{\Gamma}_i} \tilde{q}_i(y) \frac{1}{|x_k-y|} ds_y - \int_{\hat{\Gamma}_i} \tilde{v}_i(y) \frac{\partial}{\partial n} \frac{1}{|x_k-y|} ds_y \right), \quad x_k \in \hat{\Gamma}_l, \quad l = 0, 1, 2, 3, \quad (21)$$

$$-2\pi \tilde{v}_l(x_k) = \frac{\sigma_0}{\sigma_l} \int_{\hat{\Gamma}_l} \tilde{q}_l(y) \frac{1}{|x_k-y|} ds_y - \int_{\hat{\Gamma}_l} \tilde{v}_l(y) \frac{\partial}{\partial n} \frac{1}{|x_k-y|} ds_y, \quad x_k \in \hat{\Gamma}_l, \quad l = 1, 2. \quad (22)$$

Substituting (19) and (20) into (21) and (22) and changing the order of integration and summation, we obtain

$$2\pi\alpha_k + \sum_{i=0}^3 \sum_{j=M_i}^{P_i} \alpha_j \int_{\zeta_j} \frac{\partial}{\partial n} \frac{1}{|x_k - y|} ds_y = \sum_{i=0}^3 \sum_{j=M_i}^{P_i} \beta_j \int_{\zeta_j} \frac{1}{|x_k - y|} ds_y, \quad M_l \leq k \leq P_l, \quad l = 0, 1, 2, 3, \quad (23)$$

$$-2\pi\alpha_k + \sum_{j=M_l}^{P_l} \alpha_j \int_{\zeta_j} \frac{\partial}{\partial n} \frac{1}{|x_k - y|} ds_y = \frac{\sigma_0}{\sigma_l} \sum_{j=M_l}^{P_l} \beta_j \int_{\zeta_j} \frac{1}{|x_k - y|} ds_y, \quad M_l \leq k \leq P_l, \quad l = 1, 2. \quad (24)$$

Consider the matrices $H_{l,i}^+$ and $H_{l,i}^-$ ($l = 0, 1, 2, 3; i = 0, 1, 2, 3$) consisting of the elements

$$h_{kj}^\pm = \begin{cases} \int_{\zeta_j} \frac{\partial}{\partial n} \frac{1}{|x_k - y|} ds_y, & k \neq j, \\ \int_{\zeta_j} \frac{\partial}{\partial n} \frac{1}{|x_k - y|} ds_y \pm 2\pi, & k = j, \end{cases} \quad (25)$$

and the matrices $G_{l,i}$ ($l = 0, 1, 2, 3; i = 0, 1, 2, 3$) consisting of the elements

$$g_{kj} = \int_{\zeta_j} \frac{1}{|x_k - y|} ds_y, \quad (26)$$

such that $M_l \leq k \leq P_l$ and $M_i \leq j \leq P_i$. Define the vectors $\mathbf{v}_i = [\alpha_{M_i}, \alpha_{M_i+1}, \dots, \alpha_{P_i}]^T$ and $\mathbf{q}_i = [\beta_{M_i}, \beta_{M_i+1}, \dots, \beta_{P_i}]^T, i = 0, 1, 2, 3$.

With the notation introduced, system (23), (24) can be written as

$$\sum_{i=0}^3 H_{\{l,i\}}^+ \mathbf{v}_i = \sum_{i=0}^3 G_{\{l,i\}} \mathbf{q}_i, \quad l = 0, 1, 2, 3, \quad (27)$$

$$H_{\{l,l\}}^- \mathbf{v}_l = \frac{\sigma_0}{\sigma_l} G_{\{l,l\}} \mathbf{q}_l, \quad l = 1, 2. \quad (28)$$

Expressing q_1 and q_2 from (28) and substituting them into (27), we obtain a system of matrix-vector equations, which can be written as

$$\begin{aligned} H_{00}^+ \mathbf{v}_0 + R_{01} \mathbf{v}_1 + R_{02} \mathbf{v}_2 + H_{03}^+ \mathbf{v}_3 &= G_{00} \mathbf{q}_0 + G_{03} \mathbf{q}_3, \\ H_{10}^+ \mathbf{v}_0 + R_{11} \mathbf{v}_1 + R_{12} \mathbf{v}_2 + H_{13}^+ \mathbf{v}_3 &= G_{10} \mathbf{q}_0 + G_{13} \mathbf{q}_3, \\ H_{20}^+ \mathbf{v}_0 + R_{21} \mathbf{v}_1 + R_{22} \mathbf{v}_2 + H_{23}^+ \mathbf{v}_3 &= G_{20} \mathbf{q}_0 + G_{23} \mathbf{q}_3, \\ H_{30}^+ \mathbf{v}_0 + R_{31} \mathbf{v}_1 + R_{32} \mathbf{v}_2 + H_{33}^+ \mathbf{v}_3 &= G_{30} \mathbf{q}_0 + G_{33} \mathbf{q}_3, \end{aligned} \quad (29)$$

where

$$R_{li} = H_{li}^+ - \frac{\sigma_i}{\sigma_0} G_{li} G_{ii}^{-1} H_{ii}^-, \quad l = 0, 1, 2, 3, \quad i = 1, 2.$$

Using the discrete analogues of (7) and (8) yields $\mathbf{v}_0 = \hat{\psi}$ and $\mathbf{q}_0 = 0$, where $\hat{\psi}$ is a given vector. Then system (29) can be written as

$$\begin{aligned} H_{00}^+ \hat{\psi} + R_{01} \mathbf{v}_1 + R_{02} \mathbf{v}_2 + H_{03}^+ \mathbf{v}_3 &= G_{03} \mathbf{q}_3, \\ H_{10}^+ \hat{\psi} + R_{11} \mathbf{v}_1 + R_{12} \mathbf{v}_2 + H_{13}^+ \mathbf{v}_3 &= G_{13} \mathbf{q}_3, \\ H_{20}^+ \hat{\psi} + R_{21} \mathbf{v}_1 + R_{22} \mathbf{v}_2 + H_{23}^+ \mathbf{v}_3 &= G_{23} \mathbf{q}_3, \\ H_{30}^+ \hat{\psi} + R_{31} \mathbf{v}_1 + R_{32} \mathbf{v}_2 + H_{33}^+ \mathbf{v}_3 &= G_{33} \mathbf{q}_3. \end{aligned} \quad (30)$$

Sequentially expressing \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{q}_3 from the second, third, and fourth equations of the system, respectively, and substituting the results into the first equation, we obtain a system of linear algebraic equations

$$\hat{A} \mathbf{z} = \mathbf{f}, \quad (31)$$

where $\mathbf{z} = \mathbf{v}_3$ is the unknown vector, $\mathbf{f} = \hat{F} \hat{\psi}$ is a given vector, and \hat{A} and \hat{F} are given matrices.

The resulting system of linear algebraic equations can be viewed as a discrete analogue of operator equation (11). It is solved using Tikhonov regularization [6]. Let the right-hand side of system (31) be given with an error. Given an approximate right-hand side \mathbf{f}_δ and an error δ , the task is to find an approximate solution \mathbf{z}_δ of system of (31). In the construction of a regularized solution, it is important to use a priori information $\tilde{\mathbf{z}}$ about the desired potential on Γ_3 . On Γ_T , this can be the values of the potential extrapolated from Γ_E , while on Γ_H , this can be some potential values averaged over previous studies.

Consider the functional

$$M^\alpha[\mathbf{z}] = \|\hat{A} \mathbf{z} - \mathbf{f}_\delta\|^2 + \alpha \|\mathbf{z} - \tilde{\mathbf{z}}\|^2, \quad (32)$$

where α is a regularization parameter. The approximate solution \mathbf{z}_δ is defined as an element minimizing functional (32). The necessary condition for a minimum of functional (32) implies that \mathbf{z}_δ can be found by solving the system of linear algebraic equations

$$\alpha(\mathbf{z} - \tilde{\mathbf{z}}) + \hat{A}^T \hat{A} \mathbf{z} = \hat{A}^T \mathbf{f}_\delta.$$

Consider the results of some numerical experiments. In the first experiment, the above algorithm was used to solve the inverse electrocardiography problem with realistic geometry of the torso, heart, and lungs obtained via computer tomography. The total number of boundary elements was 2800, with 600, 800, and 700 elements lying on the surface of the heart, torso, and lungs, respectively. The coefficient σ_0 was set to unity, while σ_1 and σ_2 were equal to 5. The numerical experiment can be schematically described as follows.

1. The electric field potential $\varphi(x)$ produced by a quadrupole placed at the geometric center of the heart was specified on Γ_3 .
2. This potential value on Γ_3 was used to solve problem (1)–(5) and calculate the potential $\psi(x)$ on Γ_0 .
3. An error was introduced into $\psi(x)$ on Γ_0 so as to obtain a function $\psi_\delta(x)$ used to solve the inverse problem by applying the method described above.
4. The resulting approximate solution \mathbf{z}_δ on Γ_3 was compared with the exact solution $\varphi(x)$.

The following method was used to introduce an error into $\psi(x)$ on Γ_0 . The values of $\psi(x)$ at each node of boundary elements were pointwise perturbed at random so that the relative error of the difference between $\psi(x)$ and $\psi_\delta(x)$ in the Euclidean norm was equal to 2%. As a priori information, we used the electric field potential produced by a dipole placed at the geometric center of the heart. The regularization parameter α was chosen according to the discrepancy principle.

Figure 2 shows the exact potential $\tilde{\mathbf{z}} = \varphi$, the found values \mathbf{z}_δ , and the priori information $\tilde{\mathbf{z}}$ on the contour formed on the heart surface by a plane drawn through the geometric center of the heart at an angle of 30° to its vertical axis. The potential values are given in millivolts.

The goal of the second numerical experiment was to verify how the solution of the inverse electrocardiography problem is affected by chest inhomogeneities. This experiment can be schematically described as follows.

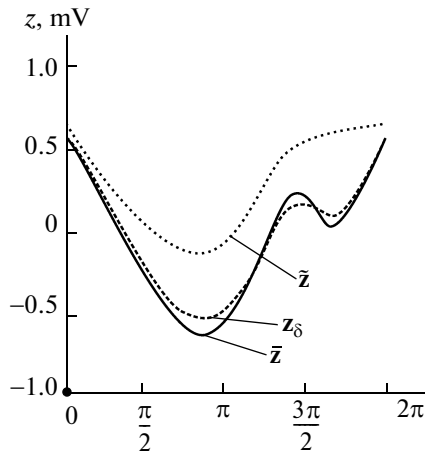


Fig. 2.

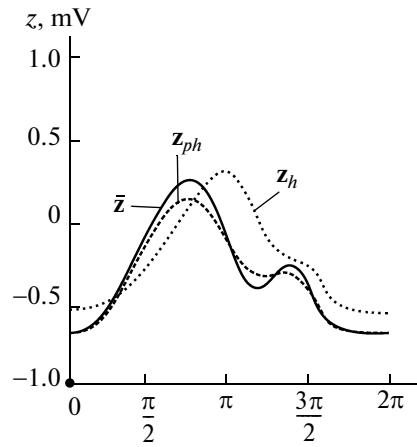


Fig. 3.

1. The electric field potential $\varphi(x)$ produced by a quadrupole placed at the geometric center of the heart was specified on Γ_3 .
2. This potential value was used to solve problem (1)–(5) and calculate the potential $\psi(x)$ on Γ_0 .
3. An error was introduced into $\psi(x)$ on Γ_0 so as to obtain a function $\psi_\delta(x)$ used to solve the inverse problems.
4. The first inverse problem was solved for the piecewise homogeneous chest model with $\sigma_0 = 1$ and $\sigma_1 = \sigma_2 = 5$.
5. The second inverse problem was solved for the homogeneous chest model with $\sigma_0 = \sigma_1 = \sigma_2 = 1$.

The geometry of the domain, the number of boundary elements, a priori information on the potential, and the method for introducing an error into $\psi(x)$ were the same as in the first experiment.

Figure 3 shows the exact potential \bar{z} and the solutions z_{ph} and z_h found with the piecewise homogeneous and homogeneous chest models, respectively, on the contour formed on the heart surface by a plane drawn through the geometric center of the heart at an angle of 120° to its vertical axis.

The results of the experiment suggest that the neglect of the chest inhomogeneity has a considerable effect on the accuracy of the solution to the inverse electrocardiography problem.

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Method of Boundary Integral Equations as Applied to the Numerical Solution of the Three-Dimensional Dirichlet Problem for the Laplace Equation in a Piecewise Homogeneous Medium

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Abstract—A Dirichlet problem is considered in a three-dimensional domain filled with a piecewise homogeneous medium. The uniqueness of its solution is proved. A system of Fredholm boundary integral equations of the second kind is constructed using the method of surface potentials, and a system of boundary integral equations of the first kind is derived directly from Green's identity. A technique for the numerical solution of integral equations is proposed, and results of numerical experiments are presented.

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Key words: Dirichlet problem for the Laplace equation, piecewise homogeneous medium, method of boundary integral equations.

1. INTRODUCTION

The conjugation problem for the Laplace equation is a classical model in the theory of direct currents in piecewise homogeneous conducting media (see [1]). It naturally arises in the theory of direct-current electrical exploration (see [2, 3]) and in the simulation of electrical engineering systems (see, e.g., [4]). This class of problems concerns to boundary value problems in unbounded domains.

In the study of bioelectric phenomena, the conjugation problem models tissues inhomogeneities and most frequently arises as an interior problem with Dirichlet or Neumann boundary conditions. Specifically, interest in Dirichlet problems for piecewise homogeneous three-dimensional domains has relatively recently arisen in computational cardiac electrophysiology. Examples of problems in this area are direct and inverse electrocardiology problems (see [5]) and the modeling of cardiac excitation based on the bidomain model equations (see [6]). The conjugation problem also arises in medical diagnostics related to the processing of electroencephalography and impedancemetry data, specifically, in impedance tomography (see [7]). Many of these issues lead to inverse problems and the design of algorithms for their solution involves the development of effective methods for solving direct problems. An example of the latter is the three-dimensional Dirichlet problem for the Laplace equation in piecewise homogeneous media.

In this paper, we give the mathematical formulation of the three-dimensional Dirichlet problem and prove the uniqueness of its solution. Additionally, a system of Fredholm integral equations of the second kind is constructed and the existence of a solution to this system and the problem itself is proved. We construct a system of integral equations of the first kind with a weak singularity in the kernel and develop numerical algorithms for its solution based on interpolation and collocations (see [8–10]). Numerical results are presented.

2. FORMULATION OF THE PROBLEM AND A UNIQUENESS THEOREM

Consider a domain $\Omega = \Omega_0 \cup \Omega_1 \cup \dots \cup \Omega_N$ in R^3 (see Fig. 1). The boundaries Γ_i of Ω_i ($i = 0, 1, \dots, N$) are sufficiently smooth (Lyapunov surfaces). The problem is formulated as follows.

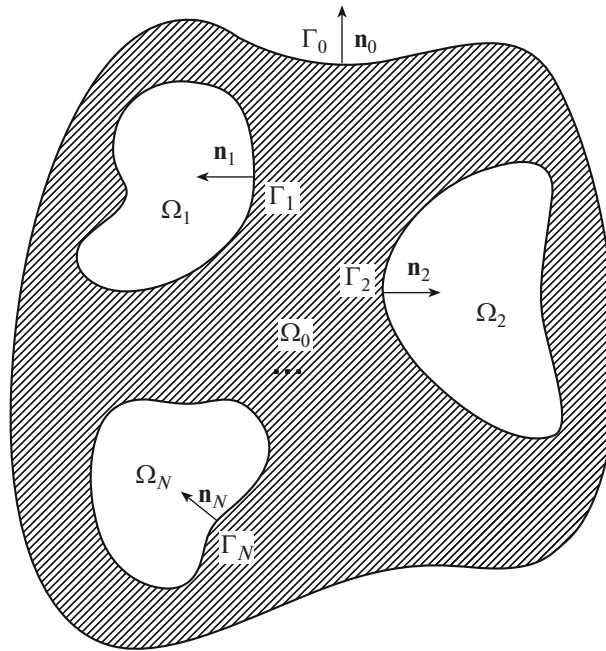


Fig. 1.

Find a function $u(x)$ such that $u \in C(\bar{\Omega})$; $u(x) = u_i(x)$, $x \in \Omega_i$, $i = 0, 1, \dots, N$, where $u_i \in C^2(\Omega_i) \cap C^1(\bar{\Omega}_i)$ and

$$\Delta u_i(x) = 0, \quad x \in \Omega_i, \quad i = 0, 1, \dots, N, \tag{1}$$

$$u_0(x) = U_0(x), \quad x \in \Gamma_0, \quad U_0(x) \in C(\Gamma_0). \tag{2}$$

The transmission conditions

$$u_0(x) = u_i(x), \quad x \in \Gamma_i, \quad i = 1, 2, \dots, N, \tag{3}$$

$$k_0 \frac{\partial u_0(x)}{\partial n} = k_i \frac{\partial u_i(x)}{\partial n}, \quad x \in \Gamma_i, \quad i = 1, 2, \dots, N. \tag{4}$$

are on Γ_i , $i = 1, 2, \dots, N$. Here, k_i ($i = 0, 1, \dots, N$) are positive and finite parameters.

Theorem 1. *The solution to problem (1)–(4) is unique.*

Proof. Let $\tilde{u}(x)$ and $\tilde{\tilde{u}}(x)$ be solutions to problem (1)–(4). Define the function

$$w(x) = \tilde{u}(x) - \tilde{\tilde{u}}(x), \quad x \in \Omega.$$

Then we have

$$\Delta w_i(x) = 0, \quad x \in \Omega_i, \quad i = 0, 1, \dots, N,$$

$$w_0(x) = 0, \quad x \in \Gamma_0.$$

On Γ_i , $i = 1, 2, \dots, N$, the following transmission conditions hold:

$$w_0(x) = w_i(x), \quad x \in \Gamma_i, \quad i = 1, 2, \dots, N, \tag{5}$$

$$k_0 \frac{\partial w_0(x)}{\partial n} = k_i \frac{\partial w_i(x)}{\partial n}, \quad x \in \Gamma_i, \quad i = 1, 2, \dots, N. \tag{6}$$

Using the first Green's identity in the multiply connected domain Ω_0 yields

$$\int_{\Omega_0} w_0 \Delta w_0 dx = \int_{\Gamma_0} w_0 \frac{\partial w_0}{\partial n} ds - \int_{\Omega_0} |\text{grad } w_0|^2 dx + \sum_{i=1}^N \int_{\Gamma_i} w_0 \frac{\partial w_0}{\partial n} ds. \tag{7}$$

Since $\Delta w_0(x) = 0$ for $x \in \Omega_0$ and $w_0(x) = 0$ for $x \in \Gamma_0$, relation (7) becomes

$$\sum_{i=1}^N \int_{\Gamma_i} w_0 \frac{\partial w_0}{\partial n} ds - \int_{\Omega_0} |\text{grad} w_0|^2 dx = 0. \quad (8)$$

For Ω_i , $i = 1, 2, \dots, N$, the Green's identity implies

$$\int_{\Omega_i} w_i \Delta w_i dx = - \int_{\Gamma_i} w_i \frac{\partial w_i}{\partial n} ds - \int_{\Omega_i} |\text{grad} w_i|^2 dx. \quad (9)$$

Since $\Delta w_i(x) = 0$ for $x \in \Omega_i$, $i = 1, 2, \dots, N$, Eq. (9) becomes

$$- \int_{\Gamma_i} w_i \frac{\partial w_i}{\partial n} ds - \int_{\Omega_i} |\text{grad} w_i|^2 dx = 0.$$

In view of (5), this can be rewritten as

$$- \int_{\Gamma_i} w_0 \frac{\partial w_i}{\partial n} ds - \int_{\Omega_i} |\text{grad} w_i|^2 dx = 0. \quad (10)$$

Equation (8) is multiplied by k_0 and each i th expression in (10) is multiplied by k_i and is added to obtain

$$\sum_{i=1}^N \int_{\Gamma_i} w_0 \left(k_0 \frac{\partial w_0}{\partial n} - k_i \frac{\partial w_i}{\partial n} \right) ds - \sum_{i=0}^N k_i \int_{\Omega_i} |\text{grad} w_i|^2 dx = 0. \quad (11)$$

In view of (6), the first sum in (11) vanishes. Thus, (11) becomes

$$\sum_{i=0}^N k_i \int_{\Omega_i} |\text{grad} w_i|^2 dx = 0. \quad (12)$$

Since $k_i > 0$, identity (12) vanishes if and only if $\text{grad} w_i(x) = 0$ in Ω_i . Therefore, $w_i(x) = \text{const}$ for $i = 0, 1, \dots, N$, and the condition $w_0(x) = 0$, $x \in \Gamma_0$, implies that $w(x) = 0$. Thus, the solution to problem (1)–(4) is unique.

3. CONSTRUCTION OF A SYSTEM OF FREDHOLM BOUNDARY INTEGRAL EQUATIONS OF THE SECOND KIND

Let $u(x)$ ($x \in \Omega$) be a solution to problem (1)–(4). Suppose that there exist functions $\mu_j(y)$, $y \in \Gamma_j$, $j = 0, 1, \dots, N$ such that $u(x)$ can be represented as

$$u(x) = \int_{\Gamma_0} \mu_0(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} ds_y + \sum_{j=1}^N \int_{\Gamma_j} \mu_j(y) \frac{k_0 - k_j}{|x-y|} ds_y, \quad (13)$$

where $|x-y|$ is the distance between the points x and y ; μ_0 is the double-layer potential density on the surface Γ_0 ; and μ_j is the single-layer potential density on Γ_j , $j = 1, 2, \dots, N$.

Note that (13) automatically satisfies conditions (1) and (3), while the fulfillment of conditions (2) and (4) leads to a system of $(N+1)$ integral equations. The first equation of the system is constructed as follows.

Let a point x be dropped from the domain Ω_0 onto the surface Γ_0 in (13). By the well-known properties of double-layer potentials, we obtain the following equation on Γ_0 :

$$2\pi\mu_0(x) + \int_{\Gamma_0} \mu_0(y) \frac{\partial}{\partial n_0} \frac{1}{|x-y|} ds_y + \sum_{j=1}^N \int_{\Gamma_j} \mu_j(y) \frac{k_0 - k_j}{|x-y|} ds_y = U_0(x), \quad x \in \Gamma_0. \quad (14)$$

Differentiating (13) along the normal n_i to Γ_i ($i = 1, 2, \dots, N$) gives N representations

$$\frac{\partial u(x)}{\partial n_i} = \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^N \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{k_0 - k_j}{|x-y|} ds_y, \quad x \in \Omega. \quad (15)$$

Let a point x be dropped from Ω_0 onto each surface Γ_i ($i = 1, 2, \dots, N$) in (15). By the properties of normal derivatives of a single-layer potential, we obtain integral equations for μ_i :

$$\frac{\partial u_0(x)}{\partial n} = (k_0 - k_i) 2\pi \mu_i(x) + \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^N \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{k_0 - k_j}{|x-y|} ds_y, \quad (16)$$

$$x \in \Gamma_i, \quad i = 1, 2, \dots, N.$$

Dropping in (15) a point x from Ω_i onto Γ_i , $i = 1, 2, \dots, N$, respectively, we obtain the integral equations

$$\frac{\partial u_i(x)}{\partial n} = -(k_0 - k_i) 2\pi \mu_i(x) + \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^N \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{k_0 - k_j}{|x-y|} ds_y, \quad (17)$$

$$x \in \Gamma_i, \quad i = 1, 2, \dots, N.$$

Subtracting (17) times $-k_i$ from (16) times k_0 and taking into account conditions (4) gives

$$2\pi(k_0 + k_i)(k_0 - k_i)\mu_i(x) + (k_0 - k_i) \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^N \left((k_0 - k_i)(k_0 - k_j) \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{1}{|x-y|} ds_y \right).$$

Thus, we have derived N integral equations

$$2\pi \mu_i(x) + \frac{1}{k_0 + k_i} \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^N \left(\frac{k_0 - k_j}{k_0 + k_i} \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{1}{|x-y|} ds_y \right) = 0, \quad i = 1, 2, \dots, N. \quad (18)$$

Combining (14) with (18), we obtain the system of integral equations

$$2\pi \mu_0(x) + \int_{\Gamma_0} \mu_0(y) \frac{\partial}{\partial n_0} \frac{1}{|x-y|} ds_y + \sum_{j=1}^N \left((k_0 - k_j) \int_{\Gamma_j} \mu_j(y) \frac{1}{|x-y|} ds_y \right) = U_0(x), \quad (19)$$

$$2\pi \mu_i(x) + \frac{1}{k_0 + k_i} \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^N \left(\frac{k_0 - k_j}{k_0 + k_i} \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{1}{|x-y|} ds_y \right) = 0, \quad i = 1, 2, \dots, N.$$

Note that (19) is a system of Fredholm integral equations of the second kind in a space of continuous functions, since the matrix of free terms is diagonal and the kernels of the system are either continuous or have a weak singularity at coinciding arguments. Therefore, the unique solvability of this system follows from the Fredholm first theorem if we show that the system of homogeneous integral equations corresponding to (19) has only the trivial solution.

Since system (19) is equivalent to the original problem, the unique solvability of (19) implies the unique solvability of problem (1)–(4).

Theorem 2. *Problem (1)–(4) has a solution.*

Proof. Consider homogeneous system (19) (with $U_0(x) \equiv 0$). Let continuous functions $\mu_0^{\text{hom}}, \dots, \mu_N^{\text{hom}}$ be a nontrivial solution to this system.

By using formula (13), homogeneous systems (19) generates a solution $u^{\text{hom}}(x)$ to homogeneous problem (1)–(4). Consider the following interior Dirichlet problems in the domains Ω_i :

$$\Delta u_i^{\text{hom}}(x) = 0, \quad x \in \Omega_i,$$

$$u_i^{\text{hom}}(x) = 0, \quad x \in \Gamma_i, \quad i = 1, 2, \dots, N.$$

If each of the functions $u_i^{\text{hom}}(x)$ is represented as a double-layer potential with the density μ_i^{hom} , then, on each boundary Γ_i , we obtain the integral equation

$$\mu_i^{\text{hom}}(x) - \frac{1}{2\pi} \int_{\Gamma_i} \mu_i^{\text{hom}}(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} ds_y = 0,$$

which has only the trivial solution. Thus, we have

$$\mu_i^{\text{hom}} = 0, \quad i = 1, 2, \dots, N.$$

Substituting μ_i^{hom} into the first equation in (19) yields an equation for μ_0^{hom} :

$$2\pi\mu_0^{\text{hom}}(x) + \int_{\Gamma_0} \mu_0^{\text{hom}}(y) \frac{\partial}{\partial n_0} \frac{1}{|x-y|} ds_y = 0,$$

which also has only the trivial solution.

Thus, the vector $\mu(x) = [\mu_0(x), \dots, \mu_N(x)]^T = 0$ solves homogeneous system (19), and, by the uniqueness theorem, there are no functions $\mu_i(x)$ that are nonzero. Therefore, the inhomogeneous system of integral equations (19) with any given function $U_0(x) \in C(\Gamma_0)$ is uniquely solvable, which proves the unique solvability of problem (1)–(4).

4. CONSTRUCTION OF A SYSTEM OF FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND AND A METHOD FOR ITS NUMERICAL SOLUTION

Along with system (19) of Fredholm integral equations of the second kind, the original differential problem (1)–(4) can be reduced to a system of integral equations of the first kind, which has a simpler structure. Specifically, the latter system can be immediately written for the function and its normal derivative and does not contain any second normal derivatives. Let us construct this system.

For the domain Ω_0 with a multiply connected boundary $\Gamma_0 \cup \dots \cup \Gamma_N$ and outward normals, using the third Green's identity and condition (3), we can write $N + 1$ equations

$$2\pi u_i(x) = \sum_{j=0}^N \left(\int_{\Gamma_j} q_j^+(y) \frac{1}{|x-y|} ds_y - \int_{\Gamma_j} u_j(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} ds_y \right), \tag{20}$$

where $i = 0, 1, \dots, N$; $x \in \Gamma_i$ are collocation points; $y \in \Gamma_j$ are integration points; $|x - y|$ is the distance between the points x and y ; and $q_j^+(y) = \partial u_0(y) / \partial n_y$.

In turn, for each domain Ω_i with a simply connected boundary Γ_i and outward normals, we can write N boundary integral equations

$$-2\pi u_i(x) = \int_{\Gamma_i} q_i^-(y) \frac{1}{|x-y|} ds_y - \int_{\Gamma_i} u_i(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} ds_y, \tag{21}$$

where $i = 1, 2, \dots, N$; $x \in \Gamma_i$ are collocation points; $y \in \Gamma_i$ are integration points; $|x - y|$ is the distance between the points x and y ; and $q_i^-(y) = \partial u_i(y) / \partial n_y$. In view of conditions (4) we have

$$q_i^-(y) = \frac{k_0}{k_i} q_i^+(y), \quad i = 1, 2, \dots, N, \quad y \in \Gamma_i,$$

and Eq. (21) can be written as

$$-2\pi u_i(x) = \int_{\Gamma_i} \frac{k_0}{k_i} q_i^+(y) \frac{1}{|x-y|} ds_y - \int_{\Gamma_i} u_i(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} ds_y, \quad i = 1, 2, \dots, N. \tag{22}$$

Combining (20) and (22) yields a system of Fredholm integral equations of the first kind

$$\begin{aligned} 2\pi u_i(x) &= \sum_{j=0}^N \left(\int_{\Gamma_j} q_j^+(y) \frac{1}{|x-y|} ds_y - \int_{\Gamma_j} u_j(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} ds_y \right), \quad x \in \Gamma_i, \quad i = 0, 1, \dots, N, \\ -2\pi u_i(x) &= \int_{\Gamma_i} \frac{k_0}{k_i} q_i^+(y) \frac{1}{|x-y|} ds_y - \int_{\Gamma_i} u_i(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} ds_y, \quad x \in \Gamma_i, \quad i = 1, 2, \dots, N. \end{aligned} \quad (23)$$

It can be shown that system (23) is equivalent to original problem (1)–(4); moreover, it is uniquely solvable if there exists a solution to problem (1)–(4), and its solution can be obtained by the method of interpolation and collocations (see [8–10]).

Following [11], we pass to a discrete representation of system (23). The surfaces Γ_i ($i = 0, 1, \dots, N$) are triangulated and each Γ_i is represented as a collection of boundary elements ds_p : $\Gamma_i = ds_1 \cup \dots \cup ds_m$. Let $\varphi_1, \dots, \varphi_m$ be a system of m linearly independent basis elements (characteristic functions) defined as

$$\varphi_p(s) = \begin{cases} 1, & s \in ds_p \\ 0, & s \notin ds_p. \end{cases}$$

The function $u(x)$ and its normal derivative are represented as expansions in terms of φ_p (piecewise constant approximation):

$$u(s) = \sum_{p=1}^m \alpha_p \varphi_p(s), \quad q(s) = \sum_{p=1}^m \beta_p \varphi_p(s), \quad (24)$$

where α_p and β_p are the values of $u(s)$ and $q(s)$, respectively, at the barycenter of the p th boundary element.

Then, after discretizing, system (23) becomes

$$\begin{aligned} 2\pi u_i &= \sum_{j=0}^N (G_{ij} q_j^+ - \hat{H}_{ij} u_j), \quad i = 0, 1, \dots, N, \\ -2\pi u_i &= \frac{k_0}{k_i} G_{ii} q_i^+ - \hat{H}_{ii} u_i, \quad i = 1, 2, \dots, N, \end{aligned} \quad (25)$$

where the matrices G_{ij} are obtained by discretizing integrals of the form

$$\int_{\Gamma_j} \frac{1}{|x-y|} ds_y, \quad x \in \Gamma_i, \quad (26)$$

and the matrices \hat{H}_{ij} are obtained by discretizing integrals of the form

$$\int_{\Gamma_j} \frac{\partial}{\partial n_y} \frac{1}{|x-y|} ds_y, \quad x \in \Gamma_i. \quad (27)$$

The matrices H_{ij}^+ and H_{ij}^- are defined as

$$\begin{aligned} H_{ij}^+ &= \begin{cases} \hat{H}_{ij}, & i \neq j \\ \hat{H}_{ij} + 2\pi E, & i = j, \end{cases} \\ H_{ij}^- &= \begin{cases} \hat{H}_{ij}, & i \neq j \\ \hat{H}_{ij} - 2\pi E, & i = j, \end{cases} \end{aligned} \quad (28)$$

where E is the identity matrix.

Table

Ω	$\frac{k_0}{k_1} = 10^{-2}$	$\frac{k_0}{k_1} = 10^{-1}$	$\frac{k_0}{k_1} = 10^0$	$\frac{k_0}{k_1} = 10^1$	$\frac{k_0}{k_1} = 10^2$
$\Omega_0 : a_0 = 3$	3.84×10^{-2}	2.93×10^{-2}	1.74×10^{-4}	1.60×10^{-2}	1.83×10^{-2}
$\Omega_1 : a = 1$	3.41×10^{-2}	2.62×10^{-2}	1.68×10^{-4}	1.41×10^{-2}	1.62×10^{-2}
$\Omega_0 : a_0 = 5$	8.22×10^{-3}	6.31×10^{-3}	6.71×10^{-5}	3.52×10^{-3}	4.03×10^{-3}
$\Omega_1 : a = 1$	4.52×10^{-3}	3.51×10^{-3}	1.20×10^{-4}	1.81×10^{-3}	2.09×10^{-3}
$\Omega_0 : a_0 = 7$	3.04×10^{-3}	2.34×10^{-3}	4.55×10^{-5}	1.26×10^{-3}	1.45×10^{-3}
$\Omega_1 : a = 1$	9.67×10^{-4}	3.64×10^{-4}	1.10×10^{-4}	3.76×10^{-4}	4.17×10^{-4}

5. SOME RESULTS OF NUMERICAL EXPERIMENTS

The numerical experiments were performed according to the following scheme. Arbitrary domains $\Omega_0, \dots, \Omega_N$ with boundaries $\Gamma_0, \dots, \Gamma_N$ were specified in a system of spline 3D simulation. A software code for automatic grid generation (see [12]) was used to produce a boundary element triangulation grid on $\Gamma_i, i = 0, 1, \dots, N$. The number of boundary elements on each surface ranged from 500 to 700. The potential of electric charges was set as a boundary condition. Next, system (33) was produced and the problem was solved numerically.

To test the computational methods and schemes and estimate their errors, we used the following classical problem with an analytical solution. Let an electric field of potential U_0 with a constant gradient $\text{grad} U_0 = \text{const}$ be given in a ball Ω of radius a_0 bounded by the sphere Γ_0 with the electrical conductivity k_0 . A ball Ω_1 of radius a with the electric conductivity k_1 bounded by the sphere Γ_1 is placed in Ω_1 (here, $a < a_0$ and the centers of the balls coincide). The goal is to find the resulting electric field potential u_0 in $\Omega_0 = \Omega \setminus \Omega_1$ and u_1 in Ω_1 .

Introducing spherical coordinates with the origin at the center of the ball and solving the Laplace equations in spherical coordinates, we can show (see [13]) that the desired potential is represented in Cartesian coordinates as

$$u(\mathbf{r}) = \text{grad} U_0 \cdot \mathbf{r} + \frac{(\sigma - 1) a^3}{(\sigma + 2) |\mathbf{r}|^3} \text{grad} U_0 \cdot \mathbf{r}, \quad \mathbf{r} \in \Omega_0, \quad (34)$$

in Ω_0 and as

$$u(\mathbf{r}) = \text{grad} U_0 \cdot \mathbf{r} + \frac{(\sigma - 1)}{(\sigma + 2)} \text{grad} U_0 \cdot \mathbf{r}, \quad \mathbf{r} \in \Omega_1, \quad (35)$$

in Ω_1 . Here, $\sigma = k_1/k_0$, $\mathbf{r} = (x, y, z)^T$ is the vector from the ball center to a given point, and a is the radius of Ω_1 .

The electric conductivity was specified as $k_0 = 1$ in Ω_0 and as $k_1 = 10^n$ ($n = -2, -1, \dots, 2$) in Ω_0 . In Cartesian coordinates, the original potential U_0 was defined as

$$U_0(\mathbf{r}) = \alpha x + \beta y + \gamma z$$

with the gradient

$$\text{grad} U_0(\mathbf{r}) = (\alpha, \beta, \gamma)^T,$$

where $\mathbf{r} = (x, y, z)^T$ and α, β , and γ are constant coefficients. Next, an analytical solution to the problem was constructed and the potentials in Ω_0 and Ω_1 were calculated by formulas (34) and (35). The results were used as reference ones for comparison with the numerical solution.

For the numerical solution, the Dirichlet condition on Γ_0 was specified by computing the potential by formulas (34) and (35) at nodes of the surface triangulation grid. The numerical solution was found using a software code developed by the authors in MatLab.

To compare the numerical solution with the analytical one, we calculated the relative error

$$\varepsilon = \|\mathbf{u}_a - \mathbf{u}_n\| / \|\mathbf{u}_a\|,$$

where \mathbf{u}_a are the values of the potential computed analytically at interior grid points in Ω and \mathbf{u}_n are the numerical values of the potential at the same points.

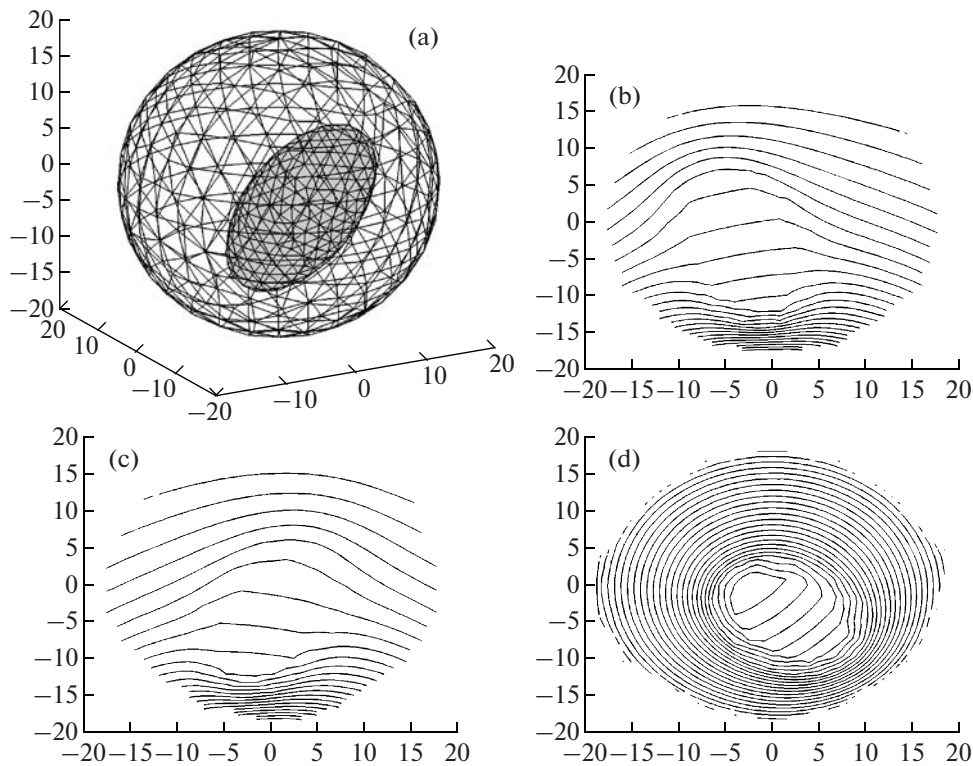


Fig. 2.

The relative errors for various radii of the spheres and various conductivities are presented in the table. These results show that the algorithm designed for the numerical solution of the Dirichlet problem for the Laplace equation in a piecewise homogeneous medium produces a solution with the relative error $\varepsilon \approx 10^{-2}-10^{-5}$.

To illustrate the capabilities of the method, we solved the Dirichlet problem for the Laplace equation in a piecewise homogeneous domain of complex geometry. A sphere Γ_0 and an ellipsoid Γ_1 with different centers were defined in a 3D surface simulation editor (see Fig. 2a). The potential of two positive electric charges located on the z axis was specified as a boundary condition. We computed the potential in the domain Ω_0 bounded by Γ_0 and Γ_1 and in the domain Ω_1 bounded by Γ_1 . The conductivities were specified as $k_0 = 1$ and $k_1 = 5$. The number of boundary elements on each surface ranged from 1400 to 1600. Figures 2b–2d show the numerical results in the form of contour lines of the potential in the planes $x = 0$, $y = 0$, and $z = 0$, respectively.

Thus, based on the methods proposed in this paper, effective algorithms can be constructed for solving three-dimensional Dirichlet problems in a piecewise homogeneous medium.

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