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NUMERICAL METHODS

Numerical Methods for Some Inverse Problems of Heart Electrophysiology

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Dedicated to the memory of Aleksandr Andreevich Samarskii, a distinguished scientist, Academician of the Russian Academy of Sciences, in honor of his ninetieth birthday

Abstract—We consider numerical methods for solving inverse problems that arise in heart electrophysiology. The first inverse problem is the Cauchy problem for the Laplace equation. Its solution algorithm is based on the Tikhonov regularization method and the method of boundary integral equations. The second inverse problem is the problem of finding the discontinuity surface of the coefficient of conductivity of a medium on the basis of the potential and its normal derivative given on the exterior surface. For its numerical solution, we suggest a method based on the method of boundary integral equations and the assumption on a special representation of the unknown surface.

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Nowadays, mathematical modeling methods are actively used in medicine. The range of problems related to the use of mathematical methods for the solution of problems of heart electrophysiology is one important direction of related research. Numerous related problems can be considered in the framework of models of stationary electric fields and reduced to various problems for equations of elliptic type. Inverse problems of heart electrophysiology arising in the processing of observation results also play an important role [1, 2]. The interpretation of these observations requires the development of stable numerical methods for inverse problems of heart electrophysiology. An important requirement imposed on these methods is that they should be capable of easy adaptation to changes in the geometry of the domains in which the problems should be solved. In the present paper, we suggest numerical methods for two inverse problems of heart electrophysiology.

1. SOLUTION METHOD FOR THE INVERSE PROBLEM OF ELECTROCARDIOGRAPHY

Consider the statement of the inverse problem of electrocardiography. Let Ω be a domain in the space R^3 bounded by a closed surface Γ from the outside and a closed surface Γ_1 from the inside. The surfaces Γ and Γ_1 are sufficiently smooth and have no common points. The surface Γ is the union of two surfaces Γ_2 and Γ_3, $\Gamma = \Gamma_2 \cup \Gamma_3$. This geometric configuration has the following interpretation: Γ_1 is an external surface of the heart, the surface Γ_2 is the human torso, and Γ_3 is the union of the top and bottom cross-sections of the torso.

The electric field of the heart is specified by sources in the cardiac muscle. Outside it, the potential of the field satisfies the Laplace equation. On the human torso, the potential is known from measurements, and its normal derivative is zero. The problem is to find the potential on the heart surface.

The mathematical statement of this inverse problem can be given in a more general form. Find a function $u(x)$ in $\bar{\Omega}$ such that

$$\Delta u(x) = 0, \qquad x \in \Omega, \tag{1}$$
$$u(x) = \varphi(x), \qquad x \in \Gamma_2, \tag{2}$$
$$\frac{\partial u(x)}{\partial n} = 0, \qquad x \in \Gamma_2, \tag{3}$$

where $\varphi(x)$ is a given function.

Problem (1)–(3) is called the Cauchy problem for the Laplace equation and is ill posed. One of the most essential manifestations of its ill-posedness is the instability of the potential $u(x)$ in $\bar{\Omega}$ under small changes in the original data $\varphi(x)$.

Numerous papers deal with the investigation of the uniqueness and the conditional stability of the Cauchy problem for the Laplace equation and the development of numerical methods for this problem (e.g., see [3–7] and the bibliography therein). The Cauchy problem for the Laplace equation as an inverse problem of electrocardiography is characterized by the fact that it is solved in a three-dimensional domain with complicated geometry. This, together with the ill-posedness of the Cauchy problem for the Laplace equation, results in substantial difficulties in the construction of numerical methods for its solution.

An algorithm for solving the Cauchy problem for the Laplace equation was suggested in [8] for the case in which the potential $u(x)$ and its normal derivative are known on the entire exterior surface Γ. In the present paper, we develop a method that permits one to solve problem (1)–(3) for the actual geometric characteristics of a human body and a real range of the potential and uses a priori information on the desired potential on the heart surface.

The Cauchy problem (1)–(3) can be stated as the problem of finding the values of the function $u(x)$ on the surfaces Γ_1 and Γ_3 under the condition that the function $u(x)$ satisfies problem (1)–(3). We denote the unknown values of $u(x)$ on Γ_1 and Γ_3 by $v(x)$ and consider the boundary value problem

$$\Delta u(x) = 0, \qquad x \in \Omega, \tag{4}$$
$$u(x) = v(x), \qquad x \in \Gamma_1 \cup \Gamma_3, \tag{5}$$
$$\frac{\partial u(x)}{\partial n} = 0, \qquad x \in \Gamma_2. \tag{6}$$

The boundary value problem (4)–(6) defines an operator A taking the values of the potential $v(x)$ on the surface $\Gamma_1 \cup \Gamma_3$ to its values $\varphi(x)$ on the surface Γ_2. The considered inverse problem is the problem of solving the operator equation of the first kind

$$Av = \varphi(x), \qquad x \in \Gamma_2, \tag{7}$$

where $v(x)$ is an unknown function and $\varphi(x)$ is a given function.

To solve Eq. (7), we use the Tikhonov regularization method [9, p. 53]. Suppose that, for the exact values $\bar{\varphi}(x)$, $x \in \Gamma_2$, there exists an exact solution $\bar{v}(x)$, $x \in \Gamma_1 \cup \Gamma_3$, of Eq. (7), but the function $\bar{\varphi}(x)$ is unknown, and we have its approximation $\varphi_\delta(x)$, $x \in \Gamma_2$, and the accuracy δ such that $\|\varphi_\delta - \bar{\varphi}\|_{L_2(\Gamma_2)} \le \delta$. The problem is to construct the approximate solution $v_\delta(x)$ on the basis of the function $\varphi_\delta(x)$ and the accuracy δ.

For the construction of a regularized solution, it is important to use information on the desired potential on the surfaces Γ_1 and Γ_3. On Γ_1 (the heart surface), these may be values of the potential on the heart surface over earlier-performed observations, and on Γ_3, these may be the values of the potential obtained after its extrapolation from the surface Γ_2. Therefore, we have some preliminary information $\tilde{v}(x)$ on the desired potential, which is naturally used in the construction of an approximate solution to increase the accuracy of computations.

Consider the functional

$$M^\alpha[v] = \|Av - \varphi_\delta\|^2_{L_2(\Gamma_2)} + \alpha\|v - \tilde{v}\|^2_{L_2(\Gamma_1 \cup \Gamma_3)}, \tag{8}$$

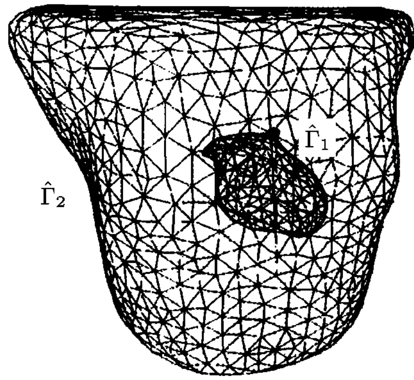

Fig. 1.

where α is a positive parameter. The approximate solution v_δ is defined as an element minimizing the functional $M^\alpha[v]$, in which the regularization parameter α depends in an appropriate way on the accuracy δ, i.e., $\alpha = \alpha(\delta)$, and can be found from the discrepancy principle

$$\|Av_\delta - \varphi_\delta\|_{L_2(\Gamma_2)} = \delta. \tag{9}$$

It follows from the necessary condition of minimum of the regularizing functional (8) that the approximate solution v_δ is a solution of the operator equation

$$\alpha(v - \tilde{v}) + A^*Av = A^*\varphi_\delta, \tag{10}$$

where the operator A^* is found from the boundary value problem [10]

$$\Delta u(x) = 0, \qquad x \in \Omega, \tag{11}$$

$$u(x) = 0, \qquad x \in \Gamma_1 \cup \Gamma_3, \tag{12}$$

$$\frac{\partial u(x)}{\partial n} = -w(x), \qquad x \in \Gamma_2. \tag{13}$$

Therefore, for the construction of an approximate solution v_δ, one should find a numerical solution of the boundary value problems (4)–(6) and (11)–(13). To this end, we use the method of boundary integral equations.

The surface $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ bounding the domain Ω is approximated by the polygonal surface $S = \hat{\Gamma}_1 \cup \hat{\Gamma}_2 \cup \hat{\Gamma}_3$, which consists of the union of N plane triangles referred to as boundary elements, $S = \zeta_1 \cup \zeta_2 \cup \cdots \cup \zeta_N$ (Fig. 1). The set of boundary elements forms a boundary element grid. The nodes of the boundary-element grid are defined as the points $x_i \in S$, $i = 1, 2, \ldots, N$, placed at the centers of gravity of the boundary elements ζ_i.

On the surface S, we introduce a system of linearly independent compactly supported basis functions $\phi_j(x)$, $x \in S$, $j = 1, 2, \ldots, N$, which are defined as follows:

$$\phi_j(x) = 1, \qquad x \in \zeta_j, \qquad \phi_j(x) = 0, \qquad x \notin \zeta_j. \tag{14}$$

Consider the approximate representation of the functions $u(x)$ and $q(x) \equiv \partial u(x)/\partial n$ in the form of the expansion in the system of basis functions $\phi_j(x)$:

$$\tilde{u}(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x), \tag{15}$$

$$\tilde{q}(x) = \sum_{j=1}^{N} \beta_j \phi_j(x), \tag{16}$$

where the expansion coefficients α_j and β_j are the values of the functions $\tilde{u}(x)$ and $\tilde{q}(x)$ at the nodes of the boundary-element grid.

For each nodal point x_i, one can write out a discrete analog of the third Green formula [11, p. 311]:

$$2\pi\tilde{u}(x_i) = \int\limits_S \tilde{q}(y)\,\frac{1}{|x_i - y|}\,ds_y - \int\limits_S \tilde{u}(y)\,\frac{\partial}{\partial n_y}\,\frac{1}{|x_i - y|}\,ds_y, \tag{17}$$

where $i = 1, 2, \ldots, N$, $x_i \in \zeta_i$, $y \in S$, and $|x_i - y|$ is the distance between the points x_i and y. By substituting the representations (15) and (16) into (17), we obtain the formula

$$2\pi\alpha_i = \int\limits_S \left(\sum_{j=1}^{N} \beta_j\phi_j(y)\right) \frac{1}{|x_i - y|}\,ds_y - \int\limits_S \left(\sum_{j=1}^{N} \alpha_j\phi_j(y)\right) \frac{\partial}{\partial n_y}\,\frac{1}{|x_i - y|}\,ds_y. \tag{18}$$

By exchanging integration and summation, we rewrite formula (18) in the form

$$2\pi\alpha_i = \sum_{j=1}^{N} \beta_j \int\limits_S \phi_j(y)\,\frac{1}{|x_i - y|}\,ds_y - \sum_{j=1}^{N} \alpha_j \int\limits_S \phi_j(y)\,\frac{\partial}{\partial n_y}\,\frac{1}{|x_i - y|}\,ds_y. \tag{19}$$

This, together with system (14), implies a system of equations for α_j and β_j ($i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$),

$$2\pi\alpha_i + \sum_{j=1}^{N} \alpha_j \int\limits_{\zeta_j} \frac{\partial}{\partial n_y}\,\frac{1}{|x_i - y|}\,ds_y = \sum_{j=1}^{N} \beta_j \int\limits_{\zeta_j} \frac{1}{|x_i - y|}\,ds_y, \tag{20}$$

which can be represented in the matrix form

$$H\mathbf{u} = G\mathbf{q}, \tag{21}$$

where H and G are the matrices computed as follows:

$$H \equiv [h_{ij}] = \begin{cases} \displaystyle\int\limits_{\zeta_j} \frac{\partial}{\partial n_y}\,\frac{1}{|x_i - y|}\,ds_y & \text{for} \quad i \neq j \\ \displaystyle\int\limits_{\zeta_j} \frac{\partial}{\partial n_y}\,\frac{1}{|x_i - y|}\,ds_y + 2\pi & \text{for} \quad i = j, \end{cases} \tag{22}$$

$$G \equiv [g_{ij}] = \int\limits_{\zeta_j} \frac{1}{|x_i - y|}\,ds_y, \tag{23}$$

$\mathbf{u} = [\alpha_1, \alpha_2, \ldots, \alpha_N]^{\mathrm{T}}$, and $\mathbf{q} = [\beta_1, \beta_2, \ldots, \beta_N]^{\mathrm{T}}$.

Consider the matrices $H_{\{2,2\}}$, $H_{\{2,13\}}$, $H_{\{13,2\}}$, and $H_{\{13,13\}}$ consisting of entries h_{ij} such that $x_i, \zeta_j \in \hat{\Gamma}_2$; $x_i \in \hat{\Gamma}_2$, $\zeta_j \in \hat{\Gamma}_1 \cup \hat{\Gamma}_3$; $x_i \in \hat{\Gamma}_1 \cup \hat{\Gamma}_3$, $\zeta_j \in \hat{\Gamma}_2$; and $x_i, \zeta_j \in \hat{\Gamma}_1 \cup \hat{\Gamma}_3$, respectively. In a similar way, we define the matrices $G_{\{2,2\}}$, $G_{\{2,13\}}$, $G_{\{13,2\}}$, and $G_{\{13,13\}}$. By $\mathbf{u}_{\{2\}}$ we denote the values α_j corresponding to the surface $\hat{\Gamma}_2$, and by $\mathbf{u}_{\{13\}}$ we denote the values α_j corresponding to the surface $\hat{\Gamma}_1 \cup \hat{\Gamma}_3$; in addition, we introduce similar notation for the vector $\mathbf{q}$.

By taking into account the above-introduced notation, one can rewrite system (21) in the form

$$\begin{bmatrix} H_{\{2,2\}} & H_{\{2,13\}} \\ H_{\{13,2\}} & H_{\{13,13\}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\{2\}} \\ \mathbf{u}_{\{13\}} \end{bmatrix} = \begin{bmatrix} G_{\{2,2\}} & G_{\{2,13\}} \\ G_{\{13,2\}} & G_{\{13,13\}} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\{2\}} \\ \mathbf{q}_{\{13\}} \end{bmatrix}. \tag{24}$$

Consider a method for the approximate solution of problem (4)–(6). Since the values of the potential on $\Gamma_1 \cup \Gamma_3$ are known and the normal derivative on Γ_2 is zero, we have $\mathbf{u}_{\{13\}} = \mathbf{v}_{\{13\}}$ and $\mathbf{q}_{\{2\}} = 0$, where $\mathbf{v}_{\{13\}}$ are given numbers. Therefore, system (24) can be rewritten in the form

$$\begin{bmatrix} H_{\{2,2\}} & H_{\{2,13\}} \\ H_{\{13,2\}} & H_{\{13,13\}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\{2\}} \\ \mathbf{v}_{\{13\}} \end{bmatrix} = \begin{bmatrix} G_{\{2,2\}} & G_{\{2,13\}} \\ G_{\{13,2\}} & G_{\{13,13\}} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{q}_{\{13\}} \end{bmatrix}. \quad (25)$$

After transformations, we obtain

$$\mathbf{u}_{\{2\}} = V_1^{-1} V_2 \mathbf{v}_{\{13\}}, \quad (26)$$

where

$$V_1 = H_{\{2,2\}} - G_{\{2,13\}} G_{\{13,13\}}^{-1} H_{\{13,2\}}, \quad (27)$$

$$V_2 = -H_{\{2,13\}} + G_{\{2,13\}} G_{\{13,13\}}^{-1} H_{\{13,13\}}. \quad (28)$$

Then the approximate solution of the boundary value problem (4)–(6) on the surface Γ_2 has the form

$$\mathbf{u}_{\{2\}} = \hat{A} \mathbf{v}_{\{13\}}, \quad (29)$$

where

$$\hat{A} = V_1^{-1} V_2. \quad (30)$$

Consider the method for the approximate solution of problem (11)–(13). Since the values of the potential are zero on $\Gamma_1 \cup \Gamma_3$ and the values of the normal derivative on Γ_2 are known, we have $\mathbf{u}_{\{13\}} = 0$ and $\mathbf{q}_{\{2\}} = -\mathbf{w}_{\{2\}}$, where the $\mathbf{w}_{\{2\}}$ are given numbers. Then system(24) can be represented in the form

$$\begin{bmatrix} H_{\{2,2\}} & H_{\{2,13\}} \\ H_{\{13,2\}} & H_{\{13,13\}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\{2\}} \\ 0 \end{bmatrix} = \begin{bmatrix} G_{\{2,2\}} & G_{\{2,13\}} \\ G_{\{13,2\}} & G_{\{13,13\}} \end{bmatrix} \begin{bmatrix} -\mathbf{w}_{\{2\}} \\ \mathbf{q}_{\{13\}} \end{bmatrix}. \quad (31)$$

After transformations, we obtain

$$\mathbf{q}_{\{13\}} = -W_1^{-1} W_2 \mathbf{w}_{\{2\}}, \quad (32)$$

where

$$W_1 = G_{\{13,13\}} - H_{\{13,2\}} H_{\{2,2\}}^{-1} G_{\{2,13\}}, \quad (33)$$

$$W_2 = -G_{\{13,2\}} + H_{\{13,2\}} H_{\{2,2\}}^{-1} G_{\{2,2\}}. \quad (34)$$

Consequently, the approximate solution of the boundary value problem (11)–(13) on the surface $\Gamma_1 \cup \Gamma_3$ has the form

$$\mathbf{q}_{\{13\}} = \hat{A}^* \mathbf{w}_{\{2\}}, \quad (35)$$

where

$$\hat{A}^* = -W_1^{-1} W_2. \quad (36)$$

Let us proceed to finding an approximate solution of Eq. (10). By (29) and (35), it can be approximated by the system of linear algebraic equations

$$\alpha(\mathbf{v}_\delta - \tilde{\mathbf{v}}) + \hat{A}^* \hat{A} \mathbf{v}_\delta = \hat{A}^* \varphi_\delta, \quad (37)$$

which can be solved with the use of the corresponding methods of computational linear algebra.

Consider an example of use of the developed method for the numerical solution of the inverse problem of electrocardiography. The actual geometric parameters of the torso and heart surfaces were found from computer tomography. The number of boundary elements on these surfaces was equal to 2500 (see Fig. 1). The scheme of the numerical experiment was the following. The potential $\bar{v}$ corresponding to a quadruple source inside the heart was posed on the surfaces Γ_1 and Γ_3. The direct problem with this potential was solved, and the function $\bar{\varphi}(x)$ was computed on Γ_2. An error was added to it, and the function $\varphi_\delta(x)$ was obtained. Then the inverse problem with

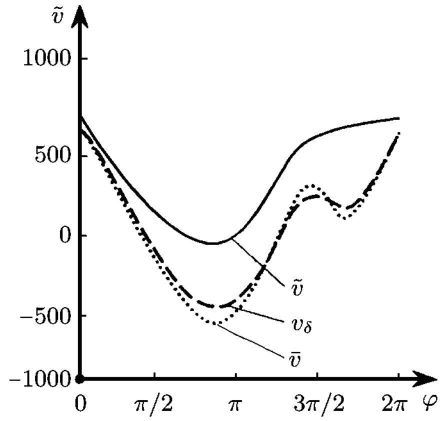

Fig. 2.

this function was solved by the suggested method, and an approximate solution v_δ was found. The function $\tilde{v}$ was chosen to be corresponding to a dipole source inside the heart. The values of the potentials $\bar{v}$, v_δ, and $\tilde{v}$ taken on a contour that represents one specific cross-section of the heart surface are shown in Fig. 2. The value of the error was $\delta = 10^{-2}$.

2. A METHOD FOR THE PROBLEM OF FINDING AN UNKNOWN SURFACE

In the investigation of processes of heart excitation, one encounters various inverse problems related to finding unknown spatial characteristics of the medium (e.g., see [2]). We consider one possible problem of such a type.

First, let us state the direct problem. Let a domain Ω in R^3 be bounded by a closed surface Γ_0, and let a closed surface Γ_1 bound a domain $\Omega_1 \subset \Omega$. Set $\Omega_0 = \Omega \backslash \bar{\Omega}_1$. The problem is to find a function $u(x)$ such that $u \in C(\bar{\Omega})$ and $u(x) = u_i(x)$, $x \in \Omega_i$ $(i = 0, 1)$, where $u_i \in C^2(\Omega_i) \cap C^1(\bar{\Omega}_i)$ and

$$\Delta u_i(x) = 0, \qquad x \in \Omega_i, \qquad i = 0, 1, \tag{38}$$

$$u_0(x) = U(x), \qquad x \in \Gamma_0, \tag{39}$$

$$u_0(x) = u_1(x), \qquad x \in \Gamma_1, \tag{40}$$

$$k_0 \frac{\partial u_0(x)}{\partial n} = k_1 \frac{\partial u_1(x)}{\partial n}, \qquad x \in \Gamma_1. \tag{41}$$

The constants k_0 and k_1 are positive.

The unique solvability of problem (38)–(41) was proved in [12].

Consider the following inverse problem. Let the surface Γ_0, the coefficients k_0 and k_1, and the function $U(x)$ be known, and let the interior surface Γ_1 be unknown. The problem is to find the surface Γ_1 if we have additional information on the solution of the boundary value problem (38)–(41),

$$\frac{\partial u_0(x)}{\partial n} = Q(x), \qquad x \in \Gamma_0. \tag{42}$$

In what follows, we assume that Γ_1 is a star-shaped surface with known center; i.e., it can be represented by some unknown function $r(\theta, \varphi)$ in the spherical coordinate system. The boundary value problem (38)–(41) defines a nonlinear operator A taking the function $r(\theta, \varphi)$ to the normal derivative of the potential on the surface Γ_0. Thus, the inverse problem is the problem of solving the nonlinear operator equation of the first kind

$$Ar = Q(x), \qquad x \in \Gamma_0. \tag{43}$$

Let us proceed to the numerical solution of the inverse problem. To this end, it is necessary to develop a method for the approximate solution of the boundary value problem (38)–(41) defining the operator A. We use the method of boundary integral equations.

We approximate the surfaces Γ_0 and Γ_1 by polygonal surfaces $\hat{\Gamma}_0$ and $\hat{\Gamma}_1$ that consist of plane triangles,

$$\hat{\Gamma}_0 = \zeta_1 \cup \zeta_2 \cup \dots \cup \zeta_M, \tag{44}$$

$$\hat{\Gamma}_1 = \zeta_{M+1} \cup \zeta_{M+2} \cup \dots \cup \zeta_N \tag{45}$$

with nodal points x_i ($i = 1, 2, \dots, N$) that are located at the centers of gravity of the corresponding elements ζ_i.

On the surface $\hat{\Gamma} = \hat{\Gamma}_0 \cup \hat{\Gamma}_1$, we introduce a system of linearly independent compactly supported basis functions $\phi_j(x)$, $x \in \hat{\Gamma}$, $j = 1, 2, \dots, N$, defined by relations (14). We approximately represent the functions $u_0(x)$, $u_1(x)$, $q_0(x) \equiv \dfrac{\partial u_0(x)}{\partial n}$, and $q_1(x) \equiv \dfrac{\partial u_1(x)}{\partial n}$ on the surface by expansions in the system of basis functions $\phi_j(x)$,

$$\tilde{u}_0(x) = \sum_{j=1}^{M} \alpha_j \phi_j(x), \qquad \tilde{q}_0(x) = \sum_{j=1}^{M} \beta_j \phi_j(x), \qquad x \in \hat{\Gamma}_0, \tag{46}$$

$$\tilde{u}_1(x) = \sum_{j=M+1}^{N} \alpha_j \phi_j(x), \qquad \tilde{q}_1(x) = \sum_{j=M+1}^{N} \beta_j \phi_j(x), \qquad x \in \hat{\Gamma}_1. \tag{47}$$

By writing out a discrete analog of the third Green formula in the domain Ω_0 for the nodal points $x_i \in \hat{\Gamma}_0$ and $x_i \in \hat{\Gamma}_1$ and by taking into account (40) and (41), we obtain the relations

$$\begin{aligned} 2\pi \tilde{u}_k(x_i) = \int\limits_{\hat{\Gamma}_0} \tilde{q}_0(y) \frac{1}{|x_i - y|}\, ds_y - \int\limits_{\hat{\Gamma}_0} \tilde{u}_0(y) \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|}\, ds_y + \frac{k_1}{k_0} \int\limits_{\hat{\Gamma}_1} \tilde{q}_1(y) \frac{1}{|x_i - y|}\, ds_y \\ - \int\limits_{\hat{\Gamma}_1} \tilde{u}_1(y) \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|}\, ds_y, \qquad x_i \in \hat{\Gamma}_k, \qquad k = 0, 1. \end{aligned} \tag{48}$$

By writing out a discrete analog of the third Green formula in the domain Ω_1 for the nodal points $x_i \in \hat{\Gamma}_1$, we obtain

$$-2\pi \tilde{u}_1(x_i) = \int\limits_{\hat{\Gamma}_1} \tilde{q}_1(y) \frac{1}{|x_i - y|}\, ds_y - \int\limits_{\hat{\Gamma}_1} \tilde{u}_1(y) \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|}\, ds_y, \qquad x_i \in \hat{\Gamma}_1. \tag{49}$$

By substituting the representations (46) and (47) into (48) and (49) and by exchanging integration and summation, we obtain the system

$$\begin{aligned} 2\pi\alpha_i + \sum_{j=1}^{M} \alpha_j \int\limits_{\zeta_j} \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|}\, ds_y + \sum_{j=M+1}^{N} \alpha_j \int\limits_{\zeta_j} \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|}\, ds_y \\ = \sum_{j=1}^{M} \beta_j \int\limits_{\zeta_j} \frac{1}{|x_i - y|}\, ds_y + \frac{k_1}{k_0} \sum_{j=M+1}^{N} \beta_j \int\limits_{\zeta_j} \frac{1}{|x_i - y|}\, ds_y, \qquad x_i \in \hat{\Gamma}_k, \qquad k = 0, 1, \\ -2\pi\alpha_i + \sum_{j=M+1}^{N} \alpha_j \int\limits_{\zeta_j} \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|}\, ds_y = \sum_{j=M+1}^{N} \beta_j \int\limits_{\zeta_j} \frac{1}{|x_i - y|}\, ds_y, \qquad x_i \in \hat{\Gamma}_1. \end{aligned} \tag{50}$$

Consider the matrices $H^+_{\{k,l\}}$, $H^-_{\{k,l\}}$, and $G_{\{k,l\}}$ ($k = 0, 1$; $l = 0, 1$) with entries

$$h^{\pm}_{ij} = \begin{cases} \int\limits_{\zeta_j} \dfrac{\partial}{\partial n_y} \dfrac{1}{|x_i - y|}\, ds_y & \text{for} \quad i \neq j \\ \int\limits_{\zeta_j} \dfrac{\partial}{\partial n_y} \dfrac{1}{|x_i - y|}\, ds_y \pm 2\pi & \text{for} \quad i = j, \end{cases} \tag{51}$$

$$g_{ij} = \int\limits_{\zeta_j} \frac{1}{|x_i - y|}\, ds_y, \tag{52}$$

respectively, such that $x_i \in \hat{\Gamma}_k$ and $\zeta_j \in \hat{\Gamma}_l$ ($i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, N$). We introduce the vectors $\mathbf{u}_{\{0\}} = [\alpha_0, \alpha_1, \ldots, \alpha_M]^{\mathrm{T}}$, $\mathbf{u}_{\{1\}} = [\alpha_{M+1}, \alpha_{M+2}, \ldots, \alpha_N]^{\mathrm{T}}$, $\mathbf{q}_{\{0\}} = [\beta_0, \beta_1, \ldots, \beta_M]^{\mathrm{T}}$, and $\mathbf{q}_{\{1\}} = [\beta_{M+1}, \beta_{M+2}, \ldots, \beta_N]^{\mathrm{T}}$.

By virtue of the above-introduced notation, system (50) can be represented in the form

$$H^+_{\{k,0\}}\mathbf{u}_{\{0\}} + H^+_{\{k,1\}}\mathbf{u}_{\{1\}} = G_{\{k,0\}}\mathbf{q}_{\{0\}} + \frac{k_1}{k_0}\, G_{\{k,1\}}\mathbf{q}_{\{1\}}, \quad k = 0, 1, \tag{53}$$

$$H^-_{\{1,1\}}\mathbf{u}_{\{1\}} = G_{\{1,1\}}\mathbf{q}_{\{1\}}. \tag{54}$$

By expressing the vector $\mathbf{q}_{\{1\}}$ from (54), one can reduce this system to the form

$$H^+_{\{k,0\}}\mathbf{u}_{\{0\}} + R_{\{k,1\}}\mathbf{u}_{\{1\}} = G_{\{k,0\}}\mathbf{q}_{\{0\}}, \qquad k = 0, 1, \tag{55}$$

where

$$R_{\{i,j\}} = H^+_{\{i,j\}} - \frac{k_j}{k_0}\, G_{\{i,j\}} G^{-1}_{\{j,j\}} H^-_{\{j,j\}}. \tag{56}$$

Since the values of the potential on the surface Γ_0 are known, we have $\mathbf{u}_{\{0\}} = \mathbf{U}$, where $\mathbf{U}$ is a known vector. Then system (55) can be represented in the form

$$G_{\{k,0\}}\mathbf{q}_{\{0\}} - R_{\{k,1\}}\mathbf{u}_{\{1\}} = H^+_{\{k,0\}}\mathbf{U}, \qquad k = 0, 1. \tag{57}$$

By solving this system in $\mathbf{q}_{\{0\}}$, we obtain

$$\mathbf{q}_{\{0\}} = \hat{A}\mathbf{U}, \tag{58}$$

where

$$\hat{A} = (G_{\{0,0\}} - R_{\{0,1\}} R^{-1}_{\{1,1\}} G_{\{1,0\}})^{-1} (H_{\{0,0\}} - R_{\{0,1\}} R^{-1}_{\{1,1\}} H_{\{1,0\}}). \tag{59}$$

Formula (58) defines an approximate solution of problem (38)–(41).

When solving the inverse problem, we assume that the unknown surface $r(\theta, \varphi)$ can be represented as

$$r(\theta, \varphi) = \sum_{n=0}^{N} \sum_{m=-n}^{n} a_{nm} Y_n^m(\theta, \varphi), \tag{60}$$

where

$$Y_n^m(\theta, \varphi) = \begin{cases} Y_n^0(\theta, \varphi) = P_n^0(\cos\theta) & \text{for} \quad m = 0 \\ Y_n^m(\theta, \varphi) = P_n^{|m|}(\cos\theta) \sin|m|\varphi & \text{for} \quad m < 0 \\ Y_n^m(\theta, \varphi) = P_n^m(\cos\theta) \cos m\varphi & \text{for} \quad m > 0, \end{cases} \tag{61}$$

the $P_n^m(x)$ are the associated Legendre functions, and the a_{nm} are unknown parameters such that $|a_{nm}| \le C_{nm}$ with given numbers C_{nm}.

Suppose that, for the exact data $\bar{Q}(x)$, the operator equation (43) has a unique solution $\bar{r}(\theta, \varphi)$ defined by a set of parameters $\bar{a}_{nm}$. However, $\bar{Q}(x)$ is an unknown function; $Q_\delta(x)$ and an error

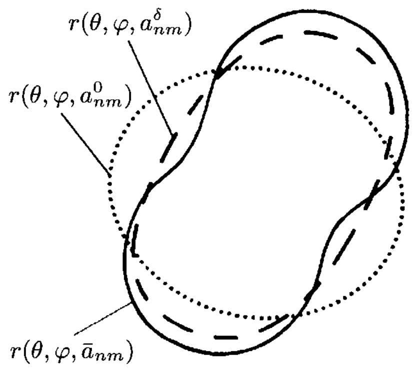

Fig. 3.

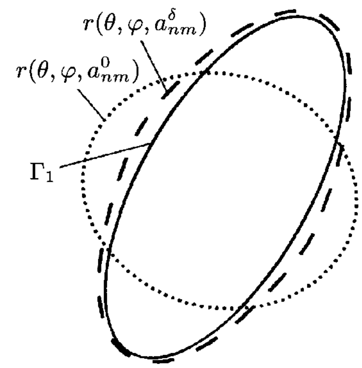

Fig. 4.

δ are given such that $\|\bar{Q} - Q_\delta\|_{L_2(\Gamma_0)} \le \delta$. In this case, an approximate solution of the inverse problem can be found by the minimization of the discrepancy functional [9, p. 38]

$$\Phi(a_{nm}) = \|Ar(\theta,\varphi;a_{nm}) - Q_\delta\|_{L_2(\Gamma_0)} \tag{62}$$

on the set $|a_{nm}| \le C_{nm}$. The validity of the inequality $\Phi(a_{nm}) \le \delta$ is a natural condition for stopping the minimization process.

Therefore, the numerical solution method for the inverse problem is a combination of an approximate computation of values of the nonlinear operator A and the minimization of the functional (62).

Let us present the results of the operation of the suggested algorithm for the solution of the inverse problem. The scheme of the numerical experiment is the following. An exact surface $r(\theta,\varphi,\bar{a}_{nm})$ given by the parameters $\bar{a}_{nm}$ was given. For that surface, we solved problem (38)–(41) and computed the normal derivative $\bar{Q}(x)$ on the surface Γ_0. Then we added an error and constructed the function $Q_\delta(x)$ for which the inverse problem was solved. Figure 3 represents the contours of cross-sections of the exact surface $r(\theta,\varphi,\bar{a}_{nm})$, the approximate surface $r(\theta,\varphi,a^{\delta}_{nm})$, and the initial approximation $r(\theta,\varphi,a^{0}_{nm})$ corresponding to $\theta = \pi/4$. The error value was $\delta = 10^{-2}$, and $N = 3$. Figure 4 represents the contours of cross-sections of the exact surface (the ellipsoid), the approximate surface $r(\theta,\varphi,a^{\delta}_{nm})$, and the initial approximation $r(\theta,\varphi,a^{0}_{nm})$ corresponding to $\theta = \pi/8$. The error value was $\delta = 10^{-2}$, and $N = 4$.

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