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Method of Boundary Integral Equations as Applied to the Numerical Solution of the Three-Dimensional Dirichlet Problem for the Laplace Equation in a Piecewise Homogeneous Medium

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Abstract—A Dirichlet problem is considered in a three-dimensional domain filled with a piecewise homogeneous medium. The uniqueness of its solution is proved. A system of Fredholm boundary integral equations of the second kind is constructed using the method of surface potentials, and a system of boundary integral equations of the first kind is derived directly from Green's identity. A technique for the numerical solution of integral equations is proposed, and results of numerical experiments are presented.

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Key words: Dirichlet problem for the Laplace equation, piecewise homogeneous medium, method of boundary integral equations.

1. INTRODUCTION

The conjugation problem for the Laplace equation is a classical model in the theory of direct currents in piecewise homogeneous conducting media (see [1]). It naturally arises in the theory of direct-current electrical exploration (see [2, 3]) and in the simulation of electrical engineering systems (see, e.g., [4]). This class of problems concerns to boundary value problems in unbounded domains.

In the study of bioelectric phenomena, the conjugation problem models tissues inhomogeneities and most frequently arises as an interior problem with Dirichlet or Neumann boundary conditions. Specifically, interest in Dirichlet problems for piecewise homogeneous three-dimensional domains has relatively recently arisen in computational cardiac electrophysiology. Examples of problems in this area are direct and inverse electrocardiology problems (see [5]) and the modeling of cardiac excitation based on the bidomain model equations (see [6]). The conjugation problem also arises in medical diagnostics related to the processing of electroencephalography and impedancemetry data, specifically, in impedance tomography (see [7]). Many of these issues lead to inverse problems and the design of algorithms for their solution involves the development of effective methods for solving direct problems. An example of the latter is the three-dimensional Dirichlet problem for the Laplace equation in piecewise homogeneous media.

In this paper, we give the mathematical formulation of the three-dimensional Dirichlet problem and prove the uniqueness of its solution. Additionally, a system of Fredholm integral equations of the second kind is constructed and the existence of a solution to this system and the problem itself is proved. We construct a system of integral equations of the first kind with a weak singularity in the kernel and develop numerical algorithms for its solution based on interpolation and collocations (see [8–10]). Numerical results are presented.

2. FORMULATION OF THE PROBLEM AND A UNIQUENESS THEOREM

Consider a domain $\Omega = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_N$ in R^3 (see Fig. 1). The boundaries Γ_i of Ω_i ($i = 0, 1, \ldots, N$) are sufficiently smooth (Lyapunov surfaces). The problem is formulated as follows.

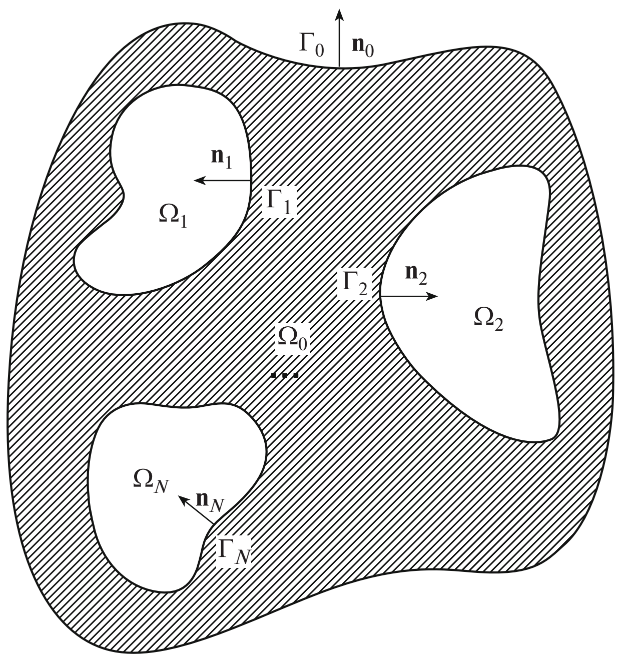

Fig. 1.

Find a function $u(x)$ such that $u \in C(\overline{\Omega})$; $u(x) = u_i(x)$, $x \in \Omega_i$, $i = 0, 1, \ldots, N$, where $u_i \in C^2(\Omega_i) \cap C^1(\overline{\Omega}_i)$ and

$$\Delta u_i(x) = 0, \quad x \in \Omega_i, \quad i = 0, 1, \ldots, N, \tag{1}$$

$$u_0(x) = U_0(x), \quad x \in \Gamma_0, \quad U_0(x) \in C(\Gamma_0). \tag{2}$$

The transmission conditions

$$u_0(x) = u_i(x), \quad x \in \Gamma_i, \quad i = 1, 2, \ldots, N, \tag{3}$$

$$k_0 \frac{\partial u_0(x)}{\partial n} = k_i \frac{\partial u_i(x)}{\partial n}, \quad x \in \Gamma_i, \quad i = 1, 2, \ldots, N. \tag{4}$$

are on Γ_i, $i = 1, 2,, \ldots, N$. Here, k_i ($i = 0, 1, \ldots, N$) are positive and finite parameters.

Theorem 1. *The solution to problem* (1)–(4) *is unique.*

Proof. Let $\tilde{u}(x)$ and $\tilde{\tilde{u}}(x)$ be solutions to problem (1)–(4). Define the function

$$w(x) = \tilde{u}(x) - \tilde{\tilde{u}}(x), \quad x \in \Omega.$$

Then we have

$$\Delta w_i(x) = 0, \quad x \in \Omega_i, \quad i = 0, 1, \ldots, N,$$

$$w_0(x) = 0, \quad x \in \Gamma_0.$$

On Γ_i, $i = 1, 2, \ldots, N$, the following transmission conditions hold:

$$w_0(x) = w_i(x), \quad x \in \Gamma_i, \quad i = 1, 2, \ldots, N, \tag{5}$$

$$k_0 \frac{\partial w_0(x)}{\partial n} = k_i \frac{\partial w_i(x)}{\partial n}, \quad x \in \Gamma_i, \quad i = 1, 2, \ldots, N. \tag{6}$$

Using the first Green's identity in the multiply connected domain Ω_0 yields

$$\int_{\Omega_0} w_0 \Delta w_0 dx = \int_{\Gamma_0} w_0 \frac{\partial w_0}{\partial n} ds - \int_{\Omega_0} |\mathrm{grad}\, w_0|^2 dx + \sum_{i=1}^{N} \int_{\Gamma_i} w_0 \frac{\partial w_0}{\partial n} ds. \tag{7}$$

Since $\Delta w_0(x) = 0$ for $x \in \Omega_0$ and $w_0(x) = 0$ for $x \in \Gamma_0$, relation (7) becomes

$$\sum_{i=1}^{N} \int_{\Gamma_i} w_0 \frac{\partial w_0}{\partial n} ds - \int_{\Omega_0} |\operatorname{grad} w_0|^2 dx = 0. \tag{8}$$

For Ω_i, $i = 1, 2, \ldots, N$, the Green's identity implies

$$\int_{\Omega_i} w_i \Delta w_i dx = -\int_{\Gamma_i} w_i \frac{\partial w_i}{\partial n} ds - \int_{\Omega_i} |\operatorname{grad} w_i|^2 dx. \tag{9}$$

Since $\Delta w_i(x) = 0$ for $x \in \Omega_i$, $i = 1, 2, \ldots, N$, Eq. (9) becomes

$$-\int_{\Gamma_i} w_i \frac{\partial w_i}{\partial n} ds - \int_{\Omega_i} |\operatorname{grad} w_i|^2 dx = 0.$$

In view of (5), this can be rewritten as

$$-\int_{\Gamma_i} w_0 \frac{\partial w_i}{\partial n} ds - \int_{\Omega_i} |\operatorname{grad} w_i|^2 dx = 0. \tag{10}$$

Equation (8) is multiplied by k_0 and each ith expression in (10) is multiplied by k_i and is added to obtain

$$\sum_{i=1}^{N} \int_{\Gamma_i} w_0 \left(k_0 \frac{\partial w_0}{\partial n} - k_i \frac{\partial w_i}{\partial n} \right) ds - \sum_{i=0}^{N} k_i \int_{\Omega_i} |\operatorname{grad} w_i|^2 dx = 0. \tag{11}$$

In view of (6), the first sum in (11) vanishes. Thus, (11) becomes

$$\sum_{i=0}^{N} k_i \int_{\Omega_i} |\operatorname{grad} w_i|^2 dx = 0. \tag{12}$$

Since $k_i > 0$, identity (12) vanishes if and only if $\operatorname{grad} w_i(x) = 0$ in Ω_i. Therefore, $w_i(x) = \text{const}$ for $i = 0, 1, \ldots, N$, and the condition $w_0(x) = 0$, $x \in \Gamma_0$, implies that $w(x) = 0$. Thus, the solution to problem (1)–(4) is unique.

3. CONSTRUCTION OF A SYSTEM OF FREDHOLM BOUNDARY INTEGRAL EQUATIONS OF THE SECOND KIND

Let $u(x)$ $(x \in \Omega)$ be a solution to problem (1)–(4). Suppose that there exist functions $\mu_j(y)$, $y \in \Gamma_j$, $j = 0, 1, \ldots, N$ such that $u(x)$ can be represented as

$$u(x) = \int_{\Gamma_0} \mu_0(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} ds_y + \sum_{j=1}^{N} \int_{\Gamma_j} \mu_j(y) \frac{k_0 - k_j}{|x-y|} ds_y, \tag{13}$$

where $|x - y|$ is the distance between the points x and y; μ_0 is the double-layer potential density on the surface Γ_0; and μ_j is the single-layer potential density on Γ_j, $j = 1, 2, \ldots, N$.

Note that (13) automatically satisfies conditions (1) and (3), while the fulfillment of conditions (2) and (4) leads to a system of $(N + 1)$ integral equations. The first equation of the system is constructed as follows.

Let a point x be dropped from the domain Ω_0 onto the surface Γ_0 in (13). By the well-known properties of double-layer potentials, we obtain the following equation on Γ_0:

$$2\pi\mu_0(x) + \int_{\Gamma_0} \mu_0(y) \frac{\partial}{\partial n_0} \frac{1}{|x-y|} ds_y + \sum_{j=1}^{N} \int_{\Gamma_j} \mu_j(y) \frac{k_0 - k_j}{|x-y|} ds_y = U_0(x), \quad x \in \Gamma_0. \tag{14}$$

Differentiating (13) along the normal n_i to Γ_i ($i = 1, 2, \ldots, N$) gives N representations

$$\frac{\partial u(x)}{\partial n_i} = \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^{N} \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{k_0 - k_j}{|x-y|} ds_y, \quad x \in \Omega. \tag{15}$$

Let a point x be dropped from Ω_0 onto each surface Γ_i ($i = 1, 2, \ldots, N$) in (15). By the properties of normal derivatives of a single-layer potential, we obtain integral equations for μ_i:

$$\frac{\partial u_0(x)}{\partial n} = (k_0 - k_i) 2\pi \mu_i(x) + \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^{N} \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{k_0 - k_j}{|x-y|} ds_y,$$
$$x \in \Gamma_i, \quad i = 1, 2, \ldots, N. \tag{16}$$

Dropping in (15) a point x from Ω_i onto Γ_i, $i = 1, 2, \ldots, N$, respectively, we obtain the integral equations

$$\frac{\partial u_i(x)}{\partial n} = -(k_0 - k_i) 2\pi \mu_i(x) + \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^{N} \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{k_0 - k_j}{|x-y|} ds_y,$$
$$x \in \Gamma_i, \quad i = 1, 2, \ldots, N. \tag{17}$$

Subtracting (17) times $-k_i$ from (16) times k_0 and taking into account conditions (4) gives

$$2\pi(k_0 + k_i)(k_0 - k_i)\mu_i(x) + (k_0 - k_i) \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y$$
$$+ \sum_{j=1}^{N} \left((k_0 - k_i)(k_0 - k_j) \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{1}{|x-y|} ds_y \right).$$

Thus, we have derived N integral equations

$$2\pi\mu_i(x) + \frac{1}{k_0 + k_i} \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^{N} \left(\frac{k_0 - k_j}{k_0 + k_i} \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{1}{|x-y|} ds_y \right) = 0, \quad i = 1, 2, \ldots, N. \tag{18}$$

Combining (14) with (18), we obtain the system of integral equations

$$2\pi\mu_0(x) + \int_{\Gamma_0} \mu_0(y) \frac{\partial}{\partial n_0} \frac{1}{|x-y|} ds_y + \sum_{j=1}^{N} \left((k_0 - k_j) \int_{\Gamma_j} \mu_j(y) \frac{1}{|x-y|} ds_y \right) = U_0(x),$$
$$2\pi\mu_i(x) + \frac{1}{k_0 + k_i} \int_{\Gamma_0} \mu_0(y) \frac{\partial^2}{\partial n_0 \partial n_i} \frac{1}{|x-y|} ds_y + \sum_{j=1}^{N} \left(\frac{k_0 - k_j}{k_0 + k_i} \int_{\Gamma_j} \mu_j(y) \frac{\partial}{\partial n_i} \frac{1}{|x-y|} ds_y \right) = 0, \quad i = 1, 2, \ldots, N. \tag{19}$$

Note that (19) is a system of Fredholm integral equations of the second kind in a space of continuous functions, since the matrix of free terms is diagonal and the kernels of the system are either continuous or have a weak singularity at coinciding arguments. Therefore, the unique solvability of this system follows from the Fredholm first theorem if we show that the system of homogeneous integral equations corresponding to (19) has only the trivial solution.

Since system (19) is equivalent to the original problem, the unique solvability of (19) implies the unique solvability of problem (1)–(4).

Theorem 2. *Problem* (1)–(4) *has a solution.*

Proof. Consider homogeneous system (19) (with $U_0(x) \equiv 0$). Let continuous functions $\mu_0^{\text{hom}}, \ldots, \mu_N^{\text{hom}}$ be a nontrivial solution to this system.

By using formula (13), homogeneous systems (19) generates a solution $u^{\text{hom}}(x)$ to homogeneous problem (1)–(4). Consider the following interior Dirichlet problems in the domains Ω_i:

$$\Delta u_i^{\text{hom}}(x) = 0, x \in \Omega_i,$$
$$u_i^{\text{hom}}(x) = 0, \quad x \in \Gamma_i, \quad i = 1, 2, \ldots, N.$$

If each of the functions $u_i^{\text{hom}}(x)$ is represented as a double-layer potential with the density μ_i^{hom}, then, on each boundary Γ_i, we obtain the integral equation

$$\mu_i^{\text{hom}}(x) - \frac{1}{2\pi}\int_{\Gamma_i}\mu_i^{\text{hom}}(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}ds_y = 0,$$

which has only the trivial solution. Thus, we have

$$\mu_i^{\text{hom}} = 0, \quad i = 1, 2, \ldots, N.$$

Substituting μ_i^{hom} into the first equation in (19) yields an equation for μ_0^{hom}:

$$2\pi\mu_0^{\text{hom}}(x) + \int_{\Gamma_0}\mu_0^{\text{hom}}(y)\frac{\partial}{\partial n_0}\frac{1}{|x-y|}ds_y = 0,$$

which also has only the trivial solution.

Thus, the vector $\mu(x) = [\mu_0(x), \ldots, \mu_N(x)]^{\text{T}} = 0$ solves homogeneous system (19), and, by the uniqueness theorem, there are no functions $\mu_i(x)$ that are nonzero. Therefore, the inhomogeneous system of integral equations (19) with any given function $U_0(x) \in C(\Gamma_0)$ is uniquely solvable, which proves the unique solvability of problem (1)–(4).

4. CONSTRUCTION OF A SYSTEM OF FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND AND A METHOD FOR ITS NUMERICAL SOLUTION

Along with system (19) of Fredholm integral equations of the second kind, the original differential problem (1)–(4) can be reduced to a system of integral equations of the first kind, which has a simpler structure. Specifically, the latter system can be immediately written for the function and its normal derivative and does not contain any second normal derivatives. Let us construct this system.

For the domain Ω_0 with a multiply connected boundary $\Gamma_0 \cup \ldots \cup \Gamma_N$ and outward normals, using the third Green's identity and condition (3), we can write $N+1$ equations

$$2\pi u_i(x) = \sum_{j=0}^{N}\left(\int_{\Gamma_j} q_j^+(y)\frac{1}{|x-y|}ds_y - \int_{\Gamma_j} u_j(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}ds_y\right), \tag{20}$$

where $i = 0, 1, \ldots, N$; $x \in \Gamma_i$ are collocation points; $y \in \Gamma_j$ are integration points; $|x-y|$ is the distance between the points x and y; and $q_j^+(y) = \partial u_0(y)/\partial n_y$.

In turn, for each domain Ω_i with a simply connected boundary Γ_i and outward normals, we can write N boundary integral equations

$$-2\pi u_i(x) = \int_{\Gamma_i} q_i^-(y)\frac{1}{|x-y|}ds_y - \int_{\Gamma_i} u_i(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}ds_y, \tag{21}$$

where $i = 1, 2, \ldots, N$; $x \in \Gamma_i$ are collocation points; $y \in \Gamma_i$ are integration points; $|x-y|$ is the distance between the points x and y; and $q_i^-(y) = \partial u_i(y)/\partial n_y$. In view of conditions (4) we have

$$q_i^-(y) = \frac{k_0}{k_i}q_i^+(y), \quad i = 1, 2, \ldots, N, \quad y \in \Gamma_i,$$

and Eq. (21) can be written as

$$-2\pi u_i(x) = \int_{\Gamma_i}\frac{k_0}{k_i}q_i^+(y)\frac{1}{|x-y|}ds_y - \int_{\Gamma_i} u_i(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}ds_y, \quad i = 1, 2, \ldots, N. \tag{22}$$

Combining (20) and (22) yields a system of Fredholm integral equations of the first kind

$$2\pi u_i(x) = \sum_{j=0}^{N}\left(\int_{\Gamma_j} q_j^+(y)\frac{1}{|x-y|}ds_y - \int_{\Gamma_j} u_j(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}ds_y\right), \quad x \in \Gamma_i, \quad i = 0, 1, \ldots, N,$$
$$-2\pi u_i(x) = \int_{\Gamma_i}\frac{k_0}{k_i} q_i^+(y)\frac{1}{|x-y|}ds_y - \int_{\Gamma_i} u_i(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}ds_y, \quad x \in \Gamma_i, \quad i = 1, 2, \ldots, N. \tag{23}$$

It can be shown that system (23) is equivalent to original problem (1)–(4); moreover, it is uniquely solvable if there exists a solution to problem (1)–(4), and its solution can be obtained by the method of interpolation and collocations (see [8–10]).

Following [11], we pass to a discrete representation of system (23). The surfaces Γ_i ($i = 0, 1, \ldots, N$) are triangulated and each Γ_i is represented as a collection of boundary elements ds_p: $\Gamma_i = ds_1 \cup \ldots \cup ds_m$. Let $\varphi_1, \ldots, \varphi_m$ be a system of m linearly independent basis elements (characteristic functions) defined as

$$\varphi_p(s) = \begin{cases} 1, & s \in ds_p \\ 0, & s \notin ds_p. \end{cases}$$

The function $u(x)$ and its normal derivative are represented as expansions in terms of φ_p (piecewise constant approximation):

$$u(s) = \sum_{p=1}^{m}\alpha_p\varphi_p(s), \quad q(s) = \sum_{p=1}^{m}\beta_p\varphi_p(s), \tag{24}$$

where α_p and β_p are the values of $u(s)$ and $q(s)$, respectively, at the barycenter of the pth boundary element.

Then, after discretizing, system (23) becomes

$$2\pi u_i = \sum_{j=0}^{N}(G_{ij}q_j^+ - \hat{H}_{ij}u_j), \quad i = 0, 1, \ldots, N,$$
$$-2\pi u_i = \frac{k_0}{k_i}G_{ii}q_i^+ - \hat{H}_{ii}u_i, \quad i = 1, 2, \ldots, N, \tag{25}$$

where the matrices G_{ij} are obtained by discretizing integrals of the form

$$\int_{\Gamma_j}\frac{1}{|x-y|}ds_y, \quad x \in \Gamma_i, \tag{26}$$

and the matrices $\hat{H}_{ij}$ are obtained by discretizing integrals of the form

$$\int_{\Gamma_j}\frac{\partial}{\partial n_y}\frac{1}{|x-y|}ds_y, \quad x \in \Gamma_i. \tag{27}$$

The matrices H_{ij}^+ and H_{ij}^- are defined as

$$H_{ij}^+ = \begin{cases} \hat{H}_{ij}, & i \neq j \\ \hat{H}_{ij} + 2\pi E, & i = j, \end{cases}$$
$$H_{ij}^- = \begin{cases} \hat{H}_{ij}, & i \neq j \\ \hat{H}_{ij} - 2\pi E, & i = j, \end{cases} \tag{28}$$

where E is the identity matrix.

Then system (25) can be written as

$$\sum_{j=0}^{N} H_{ij}^{+} u_j = \sum_{j=0}^{N} G_{ij} q_j^{+}, \quad i = 0, 1, \ldots, N,$$
$$H_{ii}^{-} u_i = \frac{k_0}{k_i} G_{ii} q_i^{+}, \quad i = 1, 2, \ldots, N. \tag{29}$$

System (29) contains $(2N + 1)$ equations and $(4N + 2)$ unknowns. A system of $(N + 1)$ equations with $(N + 1)$ unknowns can be derived from it as follows. Expressing q^+ from the second part of system (29) and substituting it into the first part of (29) yields the system

$$\begin{aligned}
R_{01}u_1 + R_{02}u_2 + \ldots + R_{0N}u_N &= G_{00}q_0^{+} - H_{00}^{+}u_0, \\
R_{11}u_1 + R_{12}u_2 + \ldots + R_{1N}u_N &= G_{10}q_0^{+} - H_{10}^{+}u_0, \\
&\ldots\ldots\ldots\ldots \\
R_{N1}u_1 + R_{N2}u_2 + \ldots + R_{NN}u_N &= G_{N0}q_0^{+} - H_{N0}^{+}u_0,
\end{aligned} \tag{30}$$

where

$$R_{ij} = H_{ij}^{+} - \frac{k_j}{k_0} G_{ij} G_{jj}^{-1} H_{jj}^{-}, \tag{31}$$

$i = 0, 1, \ldots, N$ and $j = 1, 2, \ldots, N$. Applying Dirichlet conditions (2) to (30), we obtain the system of equations

$$\begin{aligned}
-G_{00}q_0^{+} + R_{01}u_1 + R_{02}u_2 + \ldots + R_{0N}u_N &= -c_0, \\
-G_{10}q_0^{+} + R_{11}u_1 + R_{12}u_2 + \ldots + R_{1N}u_N &= -c_1, \\
&\ldots\ldots\ldots\ldots \\
-G_{N0}q_0^{+} + R_{N1}u_1 + R_{N2}u_2 + \ldots + R_{NN}u_N &= -c_N,
\end{aligned} \tag{32}$$

or, in a compact form,

$$\begin{bmatrix} -G_{00} & R_{01} & R_{02} & \ldots & R_{0N} \\ -G_{10} & R_{11} & R_{12} & \ldots & R_{1N} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -G_{N0} & R_{N1} & R_{N2} & \ldots & R_{NN} \end{bmatrix} \begin{bmatrix} q_0^{+} \\ u_1 \\ u_2 \\ \ldots \\ u_N \end{bmatrix} = \begin{bmatrix} -c_0 \\ -c_1 \\ -c_2 \\ \ldots \\ -c_N \end{bmatrix}, \tag{33}$$

where $c_i = H_{i0}^{+} U_0$.

The unknowns in the system of block matrix equations (33) can be directly expressed. However, for a large number of inhomogeneous domains, the direct expression becomes laborious. In that case, the system can be solved iteratively, for example, by the Gauss–Seidel method.

The computational algorithm consists of the following steps.

ALGORITHM

Step 1. For each boundary Γ_i, choose normal directions outward with respect to Ω_0 (see Fig. 1).

Step 2. Discretize each boundary and represent it as a triangulation grid.

Step 3. For each pair of boundaries Γ_i and Γ_j, produce the matrices G_{ij} and $\hat{H}_{ij}$ by numerically computing surface integrals (26) and (27) with collocation points $x \in \Gamma_i$ and integration points $y \in \Gamma_j$, where the points are taken at the barycenters of boundary elements.

Step 4. Calculate the matrices R_{ij} by formulas (28) and (31) with the use of Dirichlet boundary conditions. As a result, the system of block matrix equations (33) is formed.

Step 5. Solve the resulting system by directly expressing the unknowns or applying the Gauss–Seidel method.

Table

Ω	$\frac{k_0}{k_1} = 10^{-2}$	$\frac{k_0}{k_1} = 10^{-1}$	$\frac{k_0}{k_1} = 10^{0}$	$\frac{k_0}{k_1} = 10^{1}$	$\frac{k_0}{k_1} = 10^{2}$
$\Omega_0 : a_0 = 3$	3.84×10^{-2}	2.93×10^{-2}	1.74×10^{-4}	1.60×10^{-2}	1.83×10^{-2}
$\Omega_1 : a = 1$	3.41×10^{-2}	2.62×10^{-2}	1.68×10^{-4}	1.41×10^{-2}	1.62×10^{-2}
$\Omega_0 : a_0 = 5$	8.22×10^{-3}	6.31×10^{-3}	6.71×10^{-5}	3.52×10^{-3}	4.03×10^{-3}
$\Omega_1 : a = 1$	4.52×10^{-3}	3.51×10^{-3}	1.20×10^{-4}	1.81×10^{-3}	2.09×10^{-3}
$\Omega_0 : a_0 = 7$	3.04×10^{-3}	2.34×10^{-3}	4.55×10^{-5}	1.26×10^{-3}	1.45×10^{-3}
$\Omega_1 : a = 1$	9.67×10^{-4}	3.64×10^{-4}	1.10×10^{-4}	3.76×10^{-4}	4.17×10^{-4}

5. SOME RESULTS OF NUMERICAL EXPERIMENTS

The numerical experiments were performed according to the following scheme. Arbitrary domains Ω_0, ..., Ω_N with boundaries Γ_0, ..., Γ_N were specified in a system of spline 3D simulation. A software code for automatic grid generation (see [12]) was used to produce a boundary element triangulation grid on Γ_i, $i = 0, 1, ..., N$. The number of boundary elements on each surface ranged from 500 to 700. The potential of electric charges was set as a boundary condition. Next, system (33) was produced and the problem was solved numerically.

To test the computational methods and schemes and estimate their errors, we used the following classical problem with an analytical solution. Let an electric field of potential U_0 with a constant gradient $\operatorname{grad} U_0 = \text{const}$ be given in a ball Ω of radius a_0 bounded by the sphere Γ_0 with the electrical conductivity k_0. A ball Ω_1 of radius a with the electric conductivity k_1 bounded by the sphere Γ_1 is placed in Ω_1 (here, $a < a_0$ and the centers of the balls coincide). The goal is to find the resulting electric field potential u_0 in $\Omega_0 = \Omega \backslash \Omega_1$ and u_1 in Ω_1.

Introducing spherical coordinates with the origin at the center of the ball and solving the Laplace equations in spherical coordinates, we can show (see [13]) that the desired potential is represented in Cartesian coordinates as

$$u(\mathbf{r}) = \operatorname{grad} U_0 \cdot \mathbf{r} + \frac{(\sigma - 1)}{(\sigma + 2)} \frac{a^3}{|\mathbf{r}|^3} \operatorname{grad} U_0 \cdot \mathbf{r}, \quad \mathbf{r} \in \Omega_0, \tag{34}$$

in Ω_0 and as

$$u(\mathbf{r}) = \operatorname{grad} U_0 \cdot \mathbf{r} + \frac{(\sigma - 1)}{(\sigma + 2)} \operatorname{grad} U_0 \cdot \mathbf{r}, \quad \mathbf{r} \in \Omega_1, \tag{35}$$

in Ω_1. Here, $\sigma = k_1/k_0$, $\mathbf{r} = (x, y, z)^{\mathrm{T}}$ is the vector from the ball center to a given point, and a is the radius of Ω_1.

The electric conductivity was specified as $k_0 = 1$ in Ω_0 and as $k_1 = 10^n$ ($n = -2, -1, ..., 2$) in Ω_0. In Cartesian coordinates, the original potential U_0 was defined as

$$U_0(\mathbf{r}) = \alpha x + \beta y + \gamma z$$

with the gradient

$$\operatorname{grad} U_0(\mathbf{r}) = (\alpha, \beta, \gamma)^{\mathrm{T}},$$

where $\mathbf{r} = (x, y, z)^{\mathrm{T}}$ and α, β, and γ are constant coefficients. Next, an analytical solution to the problem was constructed and the potentials in Ω_0 and Ω_1 were calculated by formulas (34) and (35). The results were used as reference ones for comparison with the numerical solution.

For the numerical solution, the Dirichlet condition on Γ_0 was specified by computing the potential by formulas (34) and (35) at nodes of the surface triangulation grid. The numerical solution was found using a software code developed by the authors in MatLab.

To compare the numerical solution with the analytical one, we calculated the relative error

$$\varepsilon = \|\mathbf{u}_a - \mathbf{u}_n\| / \|\mathbf{u}_a\|,$$

where $\mathbf{u}_a$ are the values of the potential computed analytically at interior grid points in Ω and $\mathbf{u}_n$ are the numerical values of the potential at the same points.

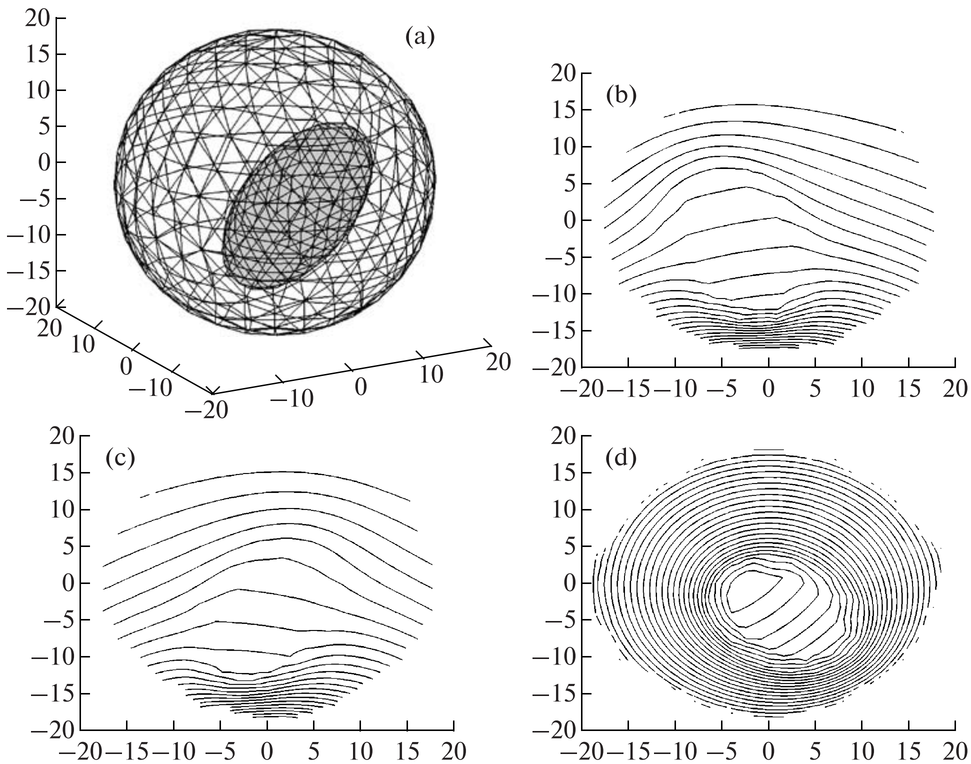

Fig. 2.

The relative errors for various radii of the spheres and various conductivities are presented in the table. These results show that the algorithm designed for the numerical solution of the Dirichlet problem for the Laplace equation in a piecewise homogeneous medium produces a solution with the relative error $\varepsilon \approx 10^{-2}$–$10^{-5}$.

To illustrate the capabilities of the method, we solved the Dirichlet problem for the Laplace equation in a piecewise homogeneous domain of complex geometry. A sphere Γ_0 and an ellipsoid Γ_1 with different centers were defined in a 3D surface simulation editor (see Fig. 2a). The potential of two positive electric charges located on the z axis was specified as a boundary condition. We computed the potential in the domain Ω_0 bounded by Γ_0 and Γ_1 and in the domain Ω_1 bounded by Γ_1. The conductivities were specified as $k_0 = 1$ and $k_1 = 5$. The number of boundary elements on each surface ranged from 1400 to 1600. Figures 2b–2d show the numerical results in the form of contour lines of the potential in the planes $x = 0$, $y = 0$, and $z = 0$, respectively.

Thus, based on the methods proposed in this paper, effective algorithms can be constructed for solving three-dimensional Dirichlet problems in a piecewise homogeneous medium.

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