Computational Mathematics and Modeling, Vol. 22, No. 1, 2011

ITERATIVE ALGORITHM FOR THE INVERSE PROBLEM OF ELECTROCARDIOGRAPHY IN A MEDIUM WITH PIECEWISE-CONSTANT ELECTRICAL CONDUCTIVITY

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UDC 517.958

The article presents a modification of the algorithm for the inverse problem of electrocardiography originally proposed in [6]. The modification is intended to improve the computation accuracy and to reduce the computing time.

Keywords: inverse problem of electrocardiography, Cauchy problem for the Laplace equation, boundary integral equations, Tikhonov regularization method, iterative algorithm.

The inverse problem of electrocardiography in potential form reconstructs the potential on the outer surface of the heart from potential measurements on the surface of the chest [1, 2]. The relevance of this inverse problem is understandable in view of the adoption in clinical practice of new techniques for the treatment of cardiac arrhythmia.

The algorithm proposed in [3] solves the inverse problem of electrocardiography for a highly schematic geometry of the trunk and the heart; a more realistic geometry is used in [4], but a homogeneous thorax is assumed; an algorithm for a piecewise-homogeneous model of the thorax has been developed in [5, 6]. In the present study we modify the algorithm of [6] with the aim of increasing the computation accuracy and reducing the computation time.

Consider the region Ω (see Fig. 1) in the space R^3, bounded from the outside by the closed surface Γ_B and from the inside by the closed surface Γ_H. The surface Γ_B is the union of two surfaces, Γ_T and Γ_E. Given are two nonintersecting regions $\Omega_i \subset \Omega$ with the boundaries Γ_i, $i = 1,2$. This geometrical configuration is amenable to the following interpretation: Γ_H is the surface of the heart, Γ_E is the part of the surface of the trunk on which the electric potential is measured, Γ_T are the upper and lower sections of the trunk, Ω_i, $i = 1,2$, are the regions of nonhomogeneity of the human thorax (the left and the right lung).

Define $\Omega_0 = \Omega \setminus (\bar{\Omega}_1 \bigcup \bar{\Omega}_1)$, $\Gamma_0 = \Gamma_E$, $\Gamma_3 = \Gamma_H \bigcup \Gamma_T$. Consider the following boundary-value problem. It is required to find the function $u(x)$, $x \in \bar{\Omega}$, such that $u(x) = u_i(x)$, $x \in \bar{\Omega}_i$, $i = 0,1,2$,

$$\Delta u_i(x) = 0, \quad x \in \Omega_i, \quad i = 0,1,2, \tag{1}$$

$$u_0(x) = \varphi(x), \quad x \in \Gamma_3, \tag{2}$$

$$\frac{\partial u_0(x)}{\partial n} = 0, \quad x \in \Gamma_0, \tag{3}$$

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Translated from Prikladnaya Matematika i Informatika, No. 34, pp. 35–40, 2010.

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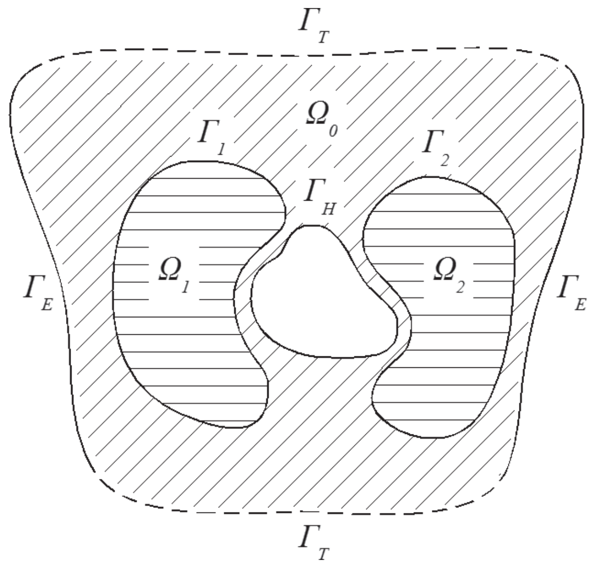

Fig. 1

$$u_0(x) = u_i(x), \quad x \in \Gamma_i, \quad i = 1,2, \tag{4}$$

$$\sigma_0 \frac{\partial u_0(x)}{\partial n} = \sigma_i \frac{\partial u_i(x)}{\partial n}, \quad x \in \Gamma_i, \quad i = 1,2. \tag{5}$$

Here $\varphi(x)$ is a given function, σ_i, $i = 1,2$, are given positive constants. The function $\varphi(x)$ is the value of the potential on the surface Γ_3 and σ_i is the electrical conductivity of the tissue filling the region Ω_i. Issues of unique solvability and numerical solution of problem (1)–(5) are considered in [5].

The inverse problem of electrocardiography can be stated as follows. Find $u(x)$, $x \in \bar{\Omega}$, such that $u(x) = u_i(x)$, $x \in \bar{\Omega}_i$, $i = 0,1,2$,

$$\Delta u_i(x) = 0, \quad x \in \Omega_i, \quad i = 0,1,2, \tag{6}$$

$$u_0(x) = \psi(x), \quad x \in \Gamma_0, \tag{7}$$

$$\frac{\partial u_0(x)}{\partial n} = 0, \quad x \in \Gamma_0, \tag{8}$$

$$u_0(x) = u_i(x), \quad x \in \Gamma_i, \quad i = 1,2, \tag{9}$$

$$\sigma_0 \frac{\partial u_0(x)}{\partial n} = \sigma_i \frac{\partial u_i(x)}{\partial n}, \quad x \in \Gamma_i, \quad i = 1,2, \tag{10}$$

where $\psi(x)$ is a known function obtained from measurements on the surface of the trunk.

Problem (6)–(10) is a generalization of the Cauchy problem for the Laplace equation and it is ill-posed. As a result of its being ill-posed, the potential $u(x)$ is unstable under small changes in the initial values $\psi(x)$. Problem (6)–(10) can be restated as the problem of finding the values of the function $u(x)$ on the surface Γ_3 given that $u(x)$ satisfies (6)–(10).

To solve the problem numerically we apply the boundary integral equation method. The main idea of the method is the following. Each surface Γ_l is approximated by a polygonal surface $\hat{\Gamma}_l$, $l = 0,1,2,3$, consisting of planar triangles. In each triangle we define a nodal point at its center of gravity. A discrete analog of the third Green's formula is written for each surface. This produces a system of relationships between the values of the functions $u_0(x)$ and $\dfrac{\partial u_0(x)}{\partial n}$ specified at the nodal points.

As a result, similarly to [6], we obtain the following system of matrix–vector equations:

$$\begin{aligned} G_{03}\mathbf{q}_3 - R_{01}\mathbf{v}_1 - R_{02}\mathbf{v}_2 - H_{03}\mathbf{v}_3 &= H_{00}\psi\,, \\ G_{13}\mathbf{q}_3 - R_{11}\mathbf{v}_1 - R_{12}\mathbf{v}_2 - H_{13}\mathbf{v}_3 &= H_{10}\psi\,, \\ G_{23}\mathbf{q}_3 - R_{21}\mathbf{v}_1 - R_{22}\mathbf{v}_2 - H_{23}\mathbf{v}_3 &= H_{20}\psi\,, \\ G_{33}\mathbf{q}_3 - R_{31}\mathbf{v}_1 - R_{32}\mathbf{v}_2 - H_{33}\mathbf{v}_3 &= H_{30}\psi\,. \end{aligned} \tag{11}$$

Here we use the following notation: the vectors $\mathbf{v}_i$ are the values of the function $u_0(x)$ at the nodal points on the surfaces $\hat{\Gamma}_i$, the vector $\mathbf{q}_3$ are the values of $\dfrac{\partial u_0(x)}{\partial n}$ at the nodal points on the surface $\hat{\Gamma}_3$; H_{ij}, R_{ij}, G_{ij} are given matrices, which are well-conditioned for $i = j$; ψ is a given vector.

An algorithm for solving the system of matrix–vector equations (11) is proposed in [6]. Successively expressing $\mathbf{v}_1$ from the second equation of the system, $\mathbf{v}_2$ from the third equation, $\mathbf{q}_3$ from the fourth equation, and substituting in the first equation, we obtain the following system of linear algebraic equations:

$$\hat{A}\mathbf{z} = \mathbf{f}\,, \tag{12}$$

where $\mathbf{z} = \mathbf{v}_3$ is an unknown vector and $\mathbf{f} = \hat{F}\psi$ is a given vector, $\hat{A}$ and $\hat{F}$ are given matrices. System (12) is solved by Tikhonov's regularization method [7].

Let us consider a different method of solving system (11). We apply the following iterative process:

$$\begin{aligned} \mathbf{q}_3^{(k+1)} &= G_{03}^{(k)}\left(R_{01}\mathbf{v}_1^{(k)} + R_{02}\mathbf{v}_2^{(k)} + H_{03}\mathbf{v}_3^{(k)} + H_{00}\psi\right), \\ \mathbf{v}_1^{(k+1)} &= R_{11}^{-1}\left(G_{13}\mathbf{q}_3^{(k+1)} - R_{12}\mathbf{v}_2^{(k)} - H_{13}\mathbf{v}_3^{(k)} - H_{10}\psi\right), \\ \mathbf{v}_2^{(k+1)} &= R_{22}^{-1}\left(G_{23}\mathbf{q}_3^{(k+1)} - R_{21}\mathbf{v}_1^{(k+1)} - H_{23}\mathbf{v}_3^{(k)} - H_{20}\psi\right), \\ \mathbf{v}_3^{(k+1)} &= H_{33}^{-1}(G_{33}\mathbf{q}_3^{(k+1)} - R_{31}\mathbf{v}_1^{(k+1)} - R_{32}\mathbf{v}_2^{(k+1)} - H_{30}\psi), \qquad k = 0,1,\ldots\,. \end{aligned} \tag{13}$$

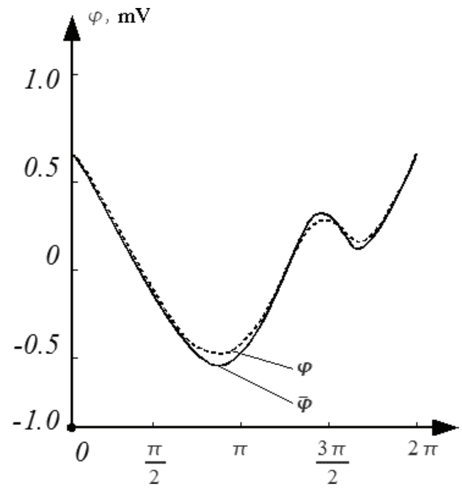

Fig. 2

The matrix $G_{03}^{(k)}$ is computed from the formula

$$G_{03}^{(k)} = (G_{03}^T G_{03} + \alpha^{(k)} E)^{-1} G_{03}^T, \tag{14}$$

where E is the identity matrix and the coefficient $\alpha^{(k)} \to 0$. The iterative process stops in step k when

$$\left\| H_{00}^{-1}\left(G_{03}\mathbf{q}_3^{(k+1)} - R_{01}\mathbf{v}_1^{(k+1)} - R_{02}\mathbf{v}_2^{(k+1)} - H_{03}\mathbf{v}_3^{(k+1)} \right) - \psi \right\| \le \delta, \tag{15}$$

where δ is a given error level.

The algorithm thus has the following form:

1. Take some initial approximations of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{q}_3$.
2. Choose some initial value of the coefficient α_0.
3. In step k compute the coefficient $\alpha^{(k)}$.
4. Evaluate the matrix $G_{03}^{(k)}$ from formula (14).
5. Compute the $(k+1)$ th approximation of the vectors $\mathbf{q}_3$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ from formula (13).
6. Check the stopping rule (15).

Let us consider some results obtained by applying the proposed algorithm for numerical solution of the inverse problem of electrocardiography assuming a realistic geometry of the trunk, the heart, and the lungs. As a test example we take the same problem as in [6]. The geometry was derived from computer tomography. The total number of boundary elements 2,800, of which 600 were on the surface of the heart, 800 on the surface of the trunk, and 700 on the surface of each lung. The coefficient σ_0 was taken equal to 1, the coefficients σ_1 and σ_2 were set equal to 5. The computer experiment proceeded according to the following scheme:

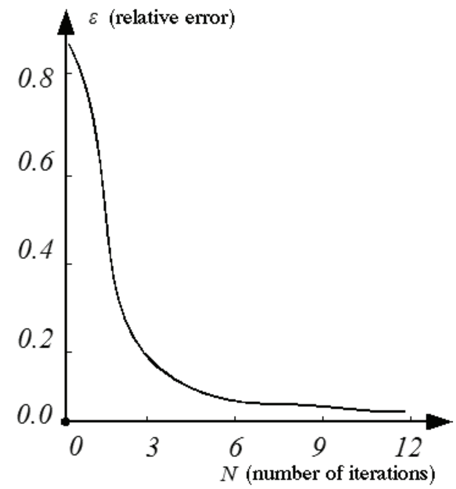

Fig. 3

1. On the surface Γ_3 specify the electric potential $\varphi(x)$ corresponding to the potential created by a quadrupole inside the heart, at its geometrical center.
2. Use this potential to solve problem (1)–(5) on the surface Γ_3 and evaluate the potential $\psi(x)$ on the surface Γ_0.
3. Inject an error into the potential $\psi(x)$ on the surface Γ_0; use the resulting function $\psi_\delta(x)$ to solve the inverse problem by the proposed method.
4. Compare the approximate solution obtained on the surface Γ_3 with the exact solution.

Figure 2 plots the exact potential $\bar{\varphi}(x)$ and the values of the computed potential $\varphi(x)$ on the contour obtained by cutting the surface of the heart with a plane passing through its geometrical center at an angle of 30 degrees to the vertical axis of the heart. Figure 3 is a graph illustrating the convergence of the iterative process.

The computer experiment shows that 9 iterations are sufficient to attain a solution with accuracy comparable to that in [6]. The iterative algorithm reduces the number of accesses and matrix multiplications, which essentially speeds up the solution of the inverse problem of electrocardiography.

The study has been supported by the Russian Foundation for Basic Research grant 08-01-00314.

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